On the global well-posedness for the Boussinesq system with horizontal dissipation

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Abstract

In this paper, we investigate the Cauchy problem for the tridimensional Boussinesq equations with horizontal dissipation. Under the assumption that the initial data is an axisymmetric without swirl, we prove the global well-posedness for this system. In the absence of vertical dissipation, there is no smoothing effect on the vertical derivatives. To make up this shortcoming, we first establish a magic relationship between $\frac{u^r}{r}$ and $\frac{\omega_\theta}{r}$ by taking full advantage of the structure of the axisymmetric fluid without swirl and some tricks in harmonic analysis. This together with the structure of the coupling of (1.2) entails the desired regularity.

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1 Introduction

The Boussinesq system describes the influence of the convection phenomenon in the dynamics of the ocean or atmosphere. In fact, it is used as a toy model for geophysical flows whenever rotation and stratification play an important role (see [28]). This system is described by the following equations:

$$\begin{cases}(\partial_t + u\cdot\nabla)u - \kappa\Delta u + \nabla p = \rho e_n, \quad (t,x)\in\mathbb{R}_+\times\mathbb{R}^n, \quad n=2,3,\\ (\partial_t + u\cdot\nabla)\rho - \nu\Delta\rho = 0,\\ \mathrm{div}u = 0,\\ (u,\rho)|_{t=0} = (u_0,\rho_0),\end{cases} \tag{1.1}$$

where, the velocity $u = (u^1,\cdots,u^n)$ is a vector field with zero divergence and ρ is a scalar quantity such as the concentration of a chemical substance or the temperature variation in a gravity fields, in which case ρe_n represents the buoyancy force. The nonnegative parameters κ and ν denote the viscosity and the molecular diffusion respectively. In addition, the pressure p is a scalar quantity which can be expressed by the unknowns u and ρ.

In the case where ν and κ are nonnegative constants, the local well-posedness of (1.1) can be easily established by using the energy method. When variables κ and ν are both positive, the classical methods allow to establish the global existence of regular solutions in dimension two and for three dimension with small initial data. Unfortunately, for the inviscid Boussinesq system (1.1), whether or not smooth solution for some nonconstant ρ_0 blows up in finite time is still an open problem. The intermediate situation has been attracted considerable attentions in the past years and important progress has been made. When ν is a positive constant and $\kappa = 0$; or $\nu = 0$ and κ is a positive constant, D. Chae [12], and T.Y. Hou and C. Li [24] proved the global well-posedness independently for the two-dimensional Boussinesq system. It is also shown the global well-posedness in the critical spaces, see [1]. In addition, C. Miao and L. Xue [27] proved the global well-posedness of the two-dimensional Boussinesq equations with fractional viscosity and thermal diffusion when the fractional powers obey mild condition. Other interesting results on the two-dimensional Boussinesq equations can be found in [4, 5, 22, 23].

Recently, there are many works devoted to the study of the tridimensional axisymmetric Boussinesq system without swirl for different viscosities. In [2], a global result was established but under some restrictive conditions on the initial density, namely it does not intersect the axis $r = 0$. Subsequently, T. Hmidi and F. Rousset [21] removed the assumption on the support of the density and proved the global well-posedness for the Navier-Stokes-Boussinesq system by virtue of the structure of the coupling between two equations of (1.1) with $\nu = 0$. In [20], they also proved the global well-posedness for the tridimensional Euler-Boussinesq system with axisymmetric initial data without swirl.

In the present paper, we consider the case that the diffusion and the viscosity only occur in the horizontal direction. More precisely,

$$\begin{cases} (\partial_t + u\cdot\nabla)u - \Delta_h u + \nabla p = \rho e_z, \quad (t,x)\in\mathbb{R}_+\times\mathbb{R}^3,\\ (\partial_t + u\cdot\nabla)\rho - \Delta_h\rho = 0,\\ \mathrm{div} u = 0,\\ (u,\rho)|_{t=0} = (u_0,\rho_0), \end{cases} \tag{1.2}$$

Here $\Delta_h = \partial_1^2+\partial_2^2$. Let us point out that the anisotropic dissipation assumption is natural in the studying of geophysical fluids. It turns out that, in certain regimes and after suitable rescaling, the vertical dissipation (or the horizontal dissipation) is negligible as compared to the horizontal dissipation (or the vertical dissipation)(see [15] for details). In fact, there are several works devoted to study of the two-dimensional Boussinesq system with anisotropic dissipation. In [17], R. Danchin and M. Paicu proved the global existence for the two-dimensional Boussinesq system with horizontal viscosity in only one equation. They mainly exhibited a polynomial control of $\|\nabla u\|_{\sqrt{L}}$, where the space $\sqrt{L}$ stands for the space of functions f in $\cap_{2\le p<\infty} L^p$ such that

$$\|f\|_{\sqrt{L}} := \sup_{2\le p<\infty} p^{-\frac12}\|f\|_{L^p} \le \infty. \tag{1.3}$$

Combining this with the following estimate

$$\|\nabla u\|_\infty \le C\big(1 + \|\nabla u\|_{\sqrt{L}}\log(e+\|u\|_{H^s})\big), \quad s>2$$

yields the global well-posedness of smooth solutions. Next, they observed the fact $\|\nabla u\|_{\sqrt{L}}$ implies that $u\in L^2_{\mathrm{loc}}(\mathbb{R}^+, \mathrm{LogLip}^{\frac12})$, where $\mathrm{LogLip}^{\frac12}$ stands for the set of bounded functions f such that

$$\sup_{x\neq y; |x-y|\le\frac12} \frac{|f(y)-f(x)|}{|x-y|\log^{\frac12}(|x-y|)^{-1}} \le +\infty. \tag{1.4}$$

And then they established the global existence with uniqueness for rough data with the help of a losing estimate. Recently, A. Adhikari, C. Cao and J. Wu also established some global results for different model under various assumption on dissipation in a series of recent papers, see in particular [4, 5, 10, 11]. In [11], C. Cao and J. Wu proved the global well-posedness for the two-dimensional Boussinesq system with vertical viscosity and vertical diffusion in terms of a Log-type inequality. In their proof, they first find that L^p-norm on vertical component of velocity with $2 \leq p < \infty$ at any time does not grow faster than $\sqrt{p\ \mathrm{log}p}$ as p increase by means of the low-high decomposition techniques.

To better understand the axisymmetric fields, let us recall some algebraic and geometric properties of the axisymmetric vector fields and discuss the special structure of the vorticity of system (1.2), see for example [19, 26]. First, we give some general statement in cylindrical coordinates: we say that a vector field u is axisymmetric if it satisfies

$$\mathcal{R}_{-\alpha}\{u(\mathcal{R}_\alpha x)\} = u(x), \quad \forall \alpha \in [0, 2\pi], \quad \forall x \in \mathbb{R}^3, \tag{1.5}$$

where $\mathcal{R}_\alpha$ denotes the rotation of axis (Oz) and with angle α. Moreover, an axisymmetric vector field u is called without swirl if it has the form:

$$u(t,x) = u^r(r,z)e_r + u^z(r,z)e_z, \quad x = (x_1, x_2, x_3), \quad r = \sqrt{x_1^2 + x_2^2} \quad \text{and } z = x_3,$$

where (e_r, e_θ, e_z) is the cylindrical basis of $\mathbb{R}^3$. Similarly, a scalar function $f : \mathbb{R}^3 \to \mathbb{R}$ is called axisymmetric if the vector field $x \mapsto f(x)e_z$ is axisymmetric, which means that

$$f(\mathcal{R}_\alpha x) = f(x), \quad \forall\, x \in \mathbb{R}^3, \quad \forall\, \alpha \in [0, 2\pi]. \tag{1.6}$$

This is equivalent to say that f depends only on r and z. Direct computations show us that the vorticity $\omega := \mathrm{curl} u$ of the vector field u takes the form

$$\omega = (\partial_z u^r - \partial_r u^z)e_\theta := \omega_\theta e_\theta.$$

On the other hand, we know that

$$u \cdot \nabla = u^r \partial_r + u^z \partial_z, \quad \mathrm{div} u = \partial_r u^r + \frac{u^r}{r} + \partial_z u^z \quad \text{and} \quad \omega \cdot \nabla u = \frac{u^r}{r}\omega \tag{1.7}$$

in the cylindrical coordinates. Therefore, the vorticity ω satisfies

$$\partial_t \omega + u \cdot \nabla \omega - \Delta_h \omega = -\partial_r \rho e_\theta + \frac{u^r}{r}\omega. \tag{1.8}$$

Since the horizontal Laplacian operator has the form $\Delta_h = \partial_{rr} + \frac{1}{r}\partial_r$ in the cylindrical coordinates then the ω_θ satisfies

$$\partial_t \omega_\theta + u \cdot \nabla \omega_\theta - \Delta_h \omega_\theta + \frac{\omega_\theta}{r^2} = -\partial_r \rho + \frac{u^r}{r}\omega_\theta. \tag{1.9}$$

In this paper, we are going to establish the global well-posedness for the system (1.2) corresponding to large axisymmetric data without swirl. Since the dissipation only occurs in the horizontal direction, it seems not obvious to get the global regularity of solutions following from [2] directly. Indeed, their proof relies on the smoothing effect on vertical direction. Also, we do not expect to obtain the growth estimate of L^p-norm about vertical component of velocity as in [11] for the tridimensional axisymmetric Boussinesq equations. Besides, as the space $H^1(\mathbb{R}^3)$ fails to be embedded in $\sqrt{L}(\mathbb{R}^3)$, it is impossible to obtain the bound of $\|\nabla u\|_{\sqrt{L}}$ in

terms of $\|\omega\|_{\sqrt{L}}$ just as in [17]. This requires us to further study the structure of axisymmetric flows and establish priori estimate to control the vorticity in $L^1_{\rm loc}(\mathbb{R}^+, L^\infty)$. Now, let us briefly to sketch the proof of results. According to (1.9) and the properties of axisymmetric flows, we find that the quantity $\frac{\omega_\theta}{r}$ satisfies

$$\big(\partial_t + u\cdot\nabla\big)\frac{\omega_\theta}{r} - \big(\Delta_h + \frac{2}{r}\partial_r\big)\frac{\omega_\theta}{r} = -\frac{\partial_r\rho}{r}. \tag{1.10}$$

We observe that the main difficulty is the lack of information about the influence of the term in the right side of (1.10) and how to use some priori estimates on ρ to control it. Therefore we need to study the properties of the operator $\frac{\partial_r}{r}$ so as to analyze the influence of the forcing term $\frac{\partial_r\rho}{r}$ on the motion of the fluid. Indeed, the behavior of $\Delta_h + \frac{\partial_r}{r}$ is like that of Δ_h, which be derived from the fact that $\frac{\partial_r}{r}$ is a part of the operator $\Delta_h = \partial_r^2 + \frac{\partial_r}{r}$. This induces us to consider the structure of the coupling between two equation of (1.2). From this observation, we introduce a new quantity $\Gamma := \frac{\omega_\theta}{r} - \frac{1}{2}\rho$ and then Γ solves the following transport equation

$$(\partial_t + u\cdot\nabla)\Gamma - (\Delta_h + \frac{2}{r}\partial_r)\Gamma = 0. \tag{1.11}$$

It follows that

$$\|\Gamma(t)\|_{L^p} \le \|\Gamma_0\|_{L^p}, \quad \forall p\in[1,\infty].$$

This together with the L^p-estimate of ρ gives that

$$\Big\|\frac{\omega_\theta}{r}(t)\Big\|_{L^p} \le \Big\|\frac{\omega_\theta}{r}(0)\Big\|_{L^p}, \quad \forall p\in[1,\infty].$$

This estimate enables us to establish a global H^1-bound of the velocity. Now, by taking the L^2-inner product of (1.9) with ω_θ and using the anisotropic inequality which will be described in Appendix A, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\rm d}{{\rm d}t}\|\omega_\theta(t)\|_{L^2}^2 + \|\nabla_h\omega_\theta(t)\|_{L^2}^2 + \Big\|\frac{\omega_\theta}{r}(t)\Big\|_{L^2}^2\\ \le&\Big\|\frac{u^r}{r}\Big\|_{L^6}^{\frac34}\Big\|\partial_z\Big(\frac{u^r}{r}\Big)\Big\|_{L^2}^{\frac14}\|\omega_\theta\|_2^{\frac12}\|\nabla_h\omega_\theta\|_2^{\frac12}\|\omega_\theta\|_2 + \|\rho\|_{L^2}^2 + \frac14\|\nabla_h\omega_\theta(t)\|_{L^2}^2 + \frac14\Big\|\frac{\omega_\theta}{r}(t)\Big\|_{L^2}^2.\end{aligned} \tag{1.12}$$

As a consequence, it is impossible to use the information of $\frac{\omega_\theta}{r}$ to control the quantity $\|\partial_z(u^r/r)\|_{L^2}$ via the following pointwise estimate established by T. Shirota and T. Yanagisawa (abbr. S-Y)

$$\Big|\frac{u^r}{r}\Big| \le C\frac{1}{|x|^2} * \Big|\frac{\omega_\theta}{r}\Big|.$$

This forces us to establish the new relationship between $\frac{u^r}{r}$ and $\frac{\omega_\theta}{r}$ instead of the S-Y estimate. To fulfill the goal, we find the following algebraic identity deduced from the geometric structure of axisymmetric flows and the Biot-Savart law:

$$\frac{u^r}{r} = \partial_z\Delta^{-1}\Big(\frac{\omega_\theta}{r}\Big) - 2\frac{\partial_r}{r}\Delta^{-1}\partial_z\Delta^{-1}\Big(\frac{\omega_\theta}{r}\Big). \tag{1.13}$$

This identity allows us to conclude the S-Y estimate, one can see Propositions 2.5 and 2.6 for more details.

Before stating our results, let us introduce the space L of those functions f which belong to every space L^p with $2\le p<\infty$ and satisfy

$$\|f\|_L := \sup_{2\le p<\infty} p^{-1}\|f\|_{L^p} \le \infty. \tag{1.14}$$

Our results are stated as follows.

Theorem 1.1. Let $u_0 \in H^1$ be an axisymmetric divergence free vector field without swirl such that $\frac{\omega_0}{r} \in L^2$ and $\partial_z \omega_0 \in L^2$. Let $\rho_0 \in H^{0,1}$ be an axisymmetric function. Then there is a unique global solution (u, ρ) of the system (1.2) such that

$$u \in \mathcal{C}(\mathbb{R}_+; H^1) \cap L^2_{\text{loc}}(\mathbb{R}_+; H^{1,2} \cap H^{2,1}), \quad \partial_z \omega \in \mathcal{C}(\mathbb{R}_+; L^2) \cap L^2_{\text{loc}}(\mathbb{R}_+; H^{1,0}),$$

$$\frac{\omega}{r} \in L^\infty_{\text{loc}}(\mathbb{R}_+; L^2) \cap L^2_{\text{loc}}(\mathbb{R}_+; H^{1,0}), \quad \rho \in \mathcal{C}(\mathbb{R}_+; H^{0,1}) \cap L^2_{\text{loc}}(\mathbb{R}_+; H^{1,1}).$$

Here and in what follows, we can refer to Section 2.1 for the definition of spaces such as H^1, $H^{0,1}$, etc.

Remark 1.1. The main difficulty is how to establish the H^1-estimates of velocity due to the lack of dissipation in the vertical direction. To overcome this difficulty, we explore an algebraic identity between $\frac{u^r}{r}$ and $\frac{\omega_\theta}{r}$, which strongly rely on the geometric structure of axisymmetric flows, and control the stretching term in vorticity equation

$$\partial_t \omega + u \cdot \nabla \omega - \Delta_h \omega = -\partial_r \rho e_\theta + \frac{u^r}{r} \omega.$$

We observe the diffusion in a direction perpendicular to the buoyancy force, and this helps us to control the source term $\partial_r \rho e_\theta$ by virtue of the horizontal smoothing effect.

Theorem 1.2. Let $u_0 \in H^1$ be an axisymmetric divergence free vector field without swirl such that $\frac{\omega_0}{r} \in L^2$ and $\omega_0 \in L^\infty$. Let $\rho_0 \in H^{0,1}$ be an axisymmetric function. Then the system (1.2) admits a unique global solution (ρ, u) such that

$$u \in \mathcal{C}_w(\mathbb{R}_+; H^1) \cap L^2_{\text{loc}}(\mathbb{R}_+; H^{1,2} \cap H^{2,1}), \quad \nabla u \in L^\infty_{\text{loc}}(\mathbb{R}_+; L),$$

$$\frac{\omega}{r} \in L^\infty_{\text{loc}}(\mathbb{R}_+; L^2) \cap L^2_{\text{loc}}(\mathbb{R}_+; H^{1,0}), \quad \rho \in \mathcal{C}_w(\mathbb{R}_+; H^{0,1}) \cap L^2_{\text{loc}}(\mathbb{R}_+; H^{1,1}) \cap \mathcal{C}_b(\mathbb{R}_+; L^2).$$

Remark 1.2. Compared with Theorem 1.1, the condition $\partial_z \omega_0 \in L^2$ has been replaced by $\omega_0 \in L^\infty$ in Theorem 1.2. It enables us to extend the global well-posedness theory to vector-field lying in space L (which ensures the vector-field belongs to LogLip space) instead of being Lipschitz. Our choice is motivated by the well-known result that the velocity in the LogLip space LL (see (6.15) in Appendix A) seems to be the minimal requirement for uniqueness to the incompressible Euler equations. Indeed, the vorticity equation can provides us the L^p-norms of vorticity with $2 \le p < \infty$. This allows us to get that $\nabla u \in L^\infty_{\text{loc}}(\mathbb{R}_+; L)$ by means of the relation $\|\nabla u\|_{L^p} \le Cp \|\omega\|_{L^p}$. Furthermore, we can obtain the global well-posedness by exploring losing estimates.

The paper is organized as follows. In Section 2 we shall give the definitions of the functional spaces that we shall use and state some useful propositions and algebraic identity. Next, we shall obtain a priori estimate for sufficiently smooth solutions of (1.2) in Section 3. The last two sections will be devoted to proving Theorems 1.1 and 1.2. In Appendix, we shall give a few technical lemmas used throughout the paper. We shall also prove an existence result and a losing estimate for the anisotropic equations with a convection term, which are the key ingredients in the proof of the results.

Notations: Throughout the paper, we write $\mathbb{R}^3 = \mathbb{R}^2_h \times \mathbb{R}_v$ The tridimensional vector field u is denoted by (u^h, u^z), and we agree that $\nabla_h = (\partial_1, \partial_2)$. Finally, the X_h (resp.,X_v) stands for that X_h is a function space over $\mathbb{R}^2_h$ (resp.,$\mathbb{R}_v$).

2 Preliminaries

2.1 Littlewood-Paley Theory and Besov spaces

In this subsection, we provide the definition of some function spaces based on the so-called Littlewood-Paley decomposition.

Let (χ,φ) be a couple of smooth functions with values in $[0,1]$ such that χ is supported in the ball $\left\{\xi\in\mathbb{R}^n \big| |\xi|\le\frac{4}{3}\right\}$, φ is supported in the shell $\left\{\xi\in\mathbb{R}^n \big| \frac{3}{4}\le|\xi|\le\frac{8}{3}\right\}$ and

$$\chi(\xi)+\sum_{j\in\mathbb{N}}\varphi(2^{-j}\xi)=1 \quad \text{for each } \xi\in\mathbb{R}^n. \tag{2.15}$$

For every $u\in\mathcal{S}'(\mathbb{R}^n)$, we define the dyadic blocks as

$$\Delta_{-1}u=\chi(D)u \quad \text{and} \quad \Delta_j u:=\varphi(2^{-j}D)u \quad \text{for each } j\in\mathbb{N}.$$

We shall also use the following low-frequency cut-off:

$$S_j u:=\chi(2^{-j}D)u.$$

One can easily show that the formal equality

$$u=\sum_{j\ge-1}\Delta_j u \tag{2.16}$$

holds in $\mathcal{S}'(\mathbb{R}^n)$, and this is called the *inhomogeneous Littlewood-Paley decomposition.* It has nice properties of quasi-orthogonality:

$$\Delta_j\Delta_{j'}u\equiv 0 \quad \text{if} \quad |j-j'|\ge 2. \tag{2.17}$$

$$\Delta_j(S_{j'-1}u\Delta_{j'}v)\equiv 0 \quad \text{if} \quad |j-j'|\ge 5. \tag{2.18}$$

Next, we first introduce the Bernstein lemma which will be useful throughout this paper.

Lemma 2.1. *There exists a constant C such that for $q,k\in\mathbb{N}$, $1\le a\le b$ and for $f\in L^a(\mathbb{R}^n)$,*

$$\begin{aligned}\sup_{|\alpha|=k}\|\partial^\alpha S_q f\|_{L^b} &\le C^k\, 2^{q(k+n(\frac{1}{a}-\frac{1}{b}))}\|S_q f\|_{L^a},\\ C^{-k}2^{qk}\|\Delta_q f\|_{L^a} &\le \sup_{|\alpha|=k}\|\partial^\alpha\Delta_q f\|_{L^a}\le C^k 2^{qk}\|\Delta_q f\|_{L^a}.\end{aligned}$$

Definition 2.1. *For $s\in\mathbb{R}$, $(p,q)\in[1,+\infty]^2$ and $u\in\mathcal{S}'(\mathbb{R}^n)$, we set*

$$\|u\|_{B^s_{p,q}(\mathbb{R}^n)}:=\Big(\sum_{j\ge-1}2^{jsq}\,\|\Delta_j u\|^q_{L^p(\mathbb{R}^n)}\Big)^{\frac{1}{q}} \quad \textit{if} \quad r<+\infty$$

and

$$\|u\|_{B^s_{p,\infty}(\mathbb{R}^n)}:=\sup_{j\ge-1}2^{js}\,\|\Delta_j u\|_{L^p(\mathbb{R}^n)}.$$

Then we define the inhomogeneous Besov spaces *as*

$$B^s_{p,q}(\mathbb{R}^n):=\left\{u\in\mathcal{S}'(\mathbb{R}^n)\big|\,\|u\|_{B^s_{p,q}(\mathbb{R}^n)}<+\infty\right\}.$$

We also denote $B^s_{2,2}$ by H^s.

Since the dissipation only occurs in the horizontal direction, it is natural to introduce the following definition.

Definition 2.2. *For $s,t\in\mathbb{R}$, $(p,q)\in[1,+\infty]^2$ and $u\in\mathcal{S}'(\mathbb{R}^3)$, we set*

$$\|u\|_{B^{s,t}_{p,q}(\mathbb{R}^3)}:=\Big(\sum_{j,k\geq -1}2^{jsq}2^{ktq}\left\|\Delta_j^h\Delta_k^v u\right\|^q_{L^p(\mathbb{R}^3)}\Big)^{\frac{1}{q}}\quad \textit{if}\quad r<+\infty$$

and

$$\|u\|_{B^{s,t}_{p,\infty}(\mathbb{R}^3)}:=\sup_{j,k\geq -1}2^{js}2^{kt}\left\|\Delta_j^h\Delta_k^v u\right\|_{L^p(\mathbb{R}^3)}.$$

Then we define the anisotropic Besov spaces *as*

$$B^{s,t}_{p,q}(\mathbb{R}^3):=\big\{u\in\mathcal{S}'(\mathbb{R}^3)\big|\ \|u\|_{B^{s,t}_{p,q}(\mathbb{R}^3)}<+\infty\big\}.$$

We also denote $B^{s,t}_{2,2}$ by $H^{s,t}$.

Let us now state some basic properties for $H^{s,t}$ spaces which will be useful later.

Lemma 2.2. *The following properties of anisotropic Besov spaces hold:*

(i) Inclusion realtion: $\|u\|_{H^{s_2,t_2}(\mathbb{R}^3)}\subseteq\|u\|_{H^{s_1,t_1}(\mathbb{R}^3)}$ if $s_2\geq s_1$ and $t_2\geq t_1$.

(ii) Interpolation: for $s_1,s_2,t_1,t_2\in\mathbb{R}$ and $\theta\in[0,1]$, we have

$$\|u\|_{H^{\theta s_1+(1-\theta)s_2,\theta t_1+(1-\theta)t_2}(\mathbb{R}^3)}\leq\|u\|^{\theta}_{H^{s_2,t_2}(\mathbb{R}^3)}\|u\|^{1-\theta}_{H^{s_1,t_1}(\mathbb{R}^3)}.$$

(iii) For $s,t\geq 0$, $\|u\|_{H^{s,t}(\mathbb{R}^3)}$ is equivalent to

$$\|u\|_{L^2(\mathbb{R}^3)}+\|\Lambda_h^s u\|_{L^2(\mathbb{R}^3)}+\left\|\Lambda_v^t u\right\|_{L^2(\mathbb{R}^3)}+\left\|\Lambda_v^t\Lambda_h^s u\right\|_{L^2(\mathbb{R}^3)}.$$

(iv) $\|u\|_{H^{s,t}(\mathbb{R}^3)}\simeq\left\|\|u\|_{H^s(\mathbb{R}^2_h)}\right\|_{H^t(\mathbb{R}_v)}\simeq\left\|\|u\|_{H^t(\mathbb{R}_v)}\right\|_{H^s(\mathbb{R}^2_h)}$.

(v) Algebraic properties: for $s>1$ and $t>\frac{1}{2}$, $\|u\|_{H^{s,t}(\mathbb{R}^3)}$ is an algebra.

Proof. We first point out that (i) and (ii) are obviously true. We only need to prove (iii), (iv) and (v). From the definition of *anisotropic Besov Spaces* $\|u\|_{H^{s,t}(\mathbb{R}^3)}$ and the Plancherel theorem, we can conclude that

$$\begin{aligned}\|u\|^2_{H^{s,t}(\mathbb{R}^3)}&=\sum_{j,k\geq -1}2^{2js}2^{2kt}\left\|\Delta_j^h\Delta_k^v u\right\|^2_{L^2(\mathbb{R}^3)}=\sum_{j,k\geq -1}2^{2js}2^{2kt}\left\|\varphi^2_{hj}\varphi^2_{vk}\hat{u}\right\|^2_{L^2(\mathbb{R}^3)}\\&=\sum_{j,k\geq -1}2^{2js}\sum_{j,k\geq -1}2^{2kt}\int_{\mathbb{R}^3}\varphi^2_{hj}\varphi^2_{vk}|\hat{u}|^2(\xi)\mathrm{d}\xi\\&=\int_{\mathbb{R}^2_h}\sum_{j\geq -1}2^{2js}\varphi^2_{hj}\int_{\mathbb{R}_v}\sum_{k\geq -1}2^{2kt}\varphi^2_{vk}|\hat{u}|^2(\xi_h,\xi_3)\mathrm{d}\xi_3\mathrm{d}\xi_h.\end{aligned}$$

This together with Equality (2.15) yields that

$$\begin{aligned}\|u\|^2_{H^{s,t}(\mathbb{R}^3)}&\simeq\int_{\mathbb{R}^3}(1+\xi_h^2)^s(1+\xi_v^2)^t|\hat{u}|^2(\xi)\mathrm{d}\xi\\&\simeq\int_{\mathbb{R}^3}|\hat{u}|^2(\xi)\mathrm{d}\xi+\int_{\mathbb{R}^3}\xi_h^{2s}|\hat{u}|^2(\xi)\mathrm{d}\xi+\int_{\mathbb{R}^3}\xi_v^{2t}|\hat{u}|^2(\xi)\mathrm{d}\xi+\int_{\mathbb{R}^3}\xi_h^{2s}\xi_v^{2t}|\hat{u}|^2(\xi)\mathrm{d}\xi\\&=\|u\|^2_{L^2(\mathbb{R}^3)}+\|\Lambda_h^s u\|^2_{L^2(\mathbb{R}^3)}+\left\|\Lambda_v^t u\right\|^2_{L^2(\mathbb{R}^3)}+\left\|\Lambda_h^s\Lambda_v^t u\right\|^2_{L^2(\mathbb{R}^3)}.\end{aligned}\tag{2.19}$$

This implies the desired result (iii).

From (2.19), it is clear that

$$\begin{aligned}\|u\|^2_{H^{s,t}(\mathbb{R}^3)} &\simeq \int_{\mathbb{R}^3}(1+\xi_h^2)^s(1+\xi_v^2)^t|\hat{u}|^2(\xi)\mathrm{d}\xi \simeq \int_{\mathbb{R}^3}|(1+\Lambda_h^s)(1+\Lambda_v^t)u|^2(x)\mathrm{d}x\\ &\simeq \left\|(1+\Lambda_v^2)^{\frac{t}{2}}\|(1+\Lambda_h^2)^{\frac{s}{2}}u\|_{L_h^2}\right\|^2_{L_v^2} \simeq \left\|\|u\|_{H^s(\mathbb{R}_h^2)}\right\|^2_{H^t(\mathbb{R}_v)}.\end{aligned}$$

Similarly, we can show that $\|u\|_{H^{s,t}(\mathbb{R}^3)} \simeq \left\|\|u\|_{H^t(\mathbb{R}_v)}\right\|_{H^s(\mathbb{R}_h^2)}$.

Finally, according to the fact that $H^s(\mathbb{R}^n)(s>\frac{n}{2})$ is an algebra and (iv). Thus, for any $u,v\in H^{s,t}(s>1,t>\frac{1}{2})$,

$$\begin{aligned}\|uv\|_{H^{s,t}} \le & C\left\|\|uv\|_{H^s(\mathbb{R}_h^2)}\right\|_{H^t(\mathbb{R}_v)} \le C\left\|\|u\|_{H^s(\mathbb{R}_h^2)}\|v\|_{H^s(\mathbb{R}_h^2)}\right\|_{H^t(\mathbb{R}_v)}\\ \le & C\left\|u\right\|_{H^{s,t}}\left\|v\right\|_{H^{s,t}}.\end{aligned}$$

This is exactly the last result. □

2.2 Heat kernel and Algebraic Identity

In this subsection, we first review the properties of the heat equation. Next, we give two useful algebraic identities and its properties.

Proposition 2.3. *[13, 25] There exist c and $C>0$ such that for every u solution of*

$$\begin{cases}\partial_t u-\Delta u=0, & x\in\mathbb{R}^n,\\ u|_{t=0}=u_0,\end{cases}$$

the following estimates hold true

- *(i)* $\|u(t)\|_{L^p(\mathbb{R}^n)} = \|e^{t\Delta}u_0\|_{L^p(\mathbb{R}^n)} \le Ct^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\|u_0\|_{L^q(\mathbb{R}^n)}$, *for* $1\le q\le p\le\infty$.
- *(ii)* $\|\Delta_j u(t)\|_{L^p(\mathbb{R}^n)} = \|e^{t\Delta}\Delta_j u_0\|_{L^p(\mathbb{R}^n)} \le Ce^{-ct2^{2j}}\|u_0\|_{L^p(\mathbb{R}^n)}$, *for* $j\ge 0$.

Next, we intend to recall the behavior of the operator $\frac{\partial_r}{r}\Delta^{-1}$ over axisymmetric functions.

Proposition 2.4. *[20] If u is an axisymmetric smooth scalar function, then we have*

$$\Big(\frac{\partial_r}{r}\Big)\Delta^{-1}u(x) = \frac{x_2^2}{r^2}\mathcal{R}_{11}u(x)+\frac{x_1^2}{r^2}\mathcal{R}_{22}u(x)-2\frac{x_1x_2}{r^2}\mathcal{R}_{12}u(x), \tag{2.20}$$

with $\mathcal{R}_{ij}=\partial_{ij}\Delta^{-1}$. Moreover, for $p\in]1,\infty[$ there exists $C>0$ such that

$$\|(\partial_r/r)\Delta^{-1}u\|_{L^p}\le C\|u\|_{L^p}. \tag{2.21}$$

Proposition 2.5. *Let u be a free divergence axisymmetric vector-field without swirl and $\omega=$ curl$u$. Then*

$$\frac{u^r}{r} = \partial_z\Delta^{-1}\left(\frac{\omega_\theta}{r}\right) - 2\frac{\partial_r}{r}\Delta^{-1}\partial_z\Delta^{-1}\left(\frac{\omega_\theta}{r}\right). \tag{2.22}$$

Besides, there hold that

$$\Big\|\partial_z\Big(\frac{u^r}{r}\Big)\Big\|_{L^p} \le C\left\|\frac{\omega_\theta}{r}\right\|_{L^p}, \quad 1<p<\infty \tag{2.23}$$

and

$$\Big\|\frac{u^r}{r}\Big\|_{L^{\frac{3q}{3-q}}} \le C\Big\|\frac{\omega_\theta}{r}\Big\|_{L^q}, \quad 1<q<3. \tag{2.24}$$

Proof. Using Biot-Savart law and the fact $\omega = \omega_\theta e_\theta$, we have

$$u^1 = \Delta^{-1}\left((\partial_z \omega_\theta)\cos\theta\right) = \partial_z \Delta^{-1}\left(x_1 \frac{\omega_\theta}{r}\right) \tag{2.25}$$

and

$$u^2 = \Delta^{-1}\left((\partial_z \omega_\theta)\sin\theta\right) = \partial_z \Delta^{-1}\left(x_2 \frac{\omega_\theta}{r}\right). \tag{2.26}$$

On the other hand, we observe that

$$\Delta^{-1}\left(x_i \frac{\omega_\theta}{r}\right) = x_i \Delta^{-1}\left(\frac{\omega_\theta}{r}\right) - \left[x_i, \Delta^{-1}\right]\left(\frac{\omega_\theta}{r}\right), \quad \text{for } i = 1, 2. \tag{2.27}$$

Applying the Laplace operator to the commutator $\left[x_i, \Delta^{-1}\right]\left(\frac{\omega_\theta}{r}\right)$, we get

$$\begin{aligned}
\Delta\left[x_i, \Delta^{-1}\right]\left(\frac{\omega_\theta}{r}\right) &= -\, x_i \frac{\omega_\theta}{r} + \Delta\left(x_i \Delta^{-1}\left(\frac{\omega_\theta}{r}\right)\right) \\
&= -\, x_i \frac{\omega_\theta}{r} + x_i \frac{\omega_\theta}{r} + 2\partial_i x_i \partial_i \Delta^{-1}\left(\frac{\omega_\theta}{r}\right) \\
&= 2\partial_i \Delta^{-1}\left(\frac{\omega_\theta}{r}\right).
\end{aligned}$$

This means that

$$\left[x_i, \Delta^{-1}\right]\left(\frac{\omega_\theta}{r}\right) = 2\partial_i \Delta^{-2}\left(\frac{\omega_\theta}{r}\right) = 2x_i \frac{\partial_r}{r}\Delta^{-2}\left(\frac{\omega_\theta}{r}\right).$$

Inserting this estimate in (2.27) gives

$$\Delta^{-1}\left(x_i \frac{\omega_\theta}{r}\right) = x_i \Delta^{-1}\left(\frac{\omega_\theta}{r}\right) - 2x_i \frac{\partial_r}{r}\Delta^{-2}\left(\frac{\omega_\theta}{r}\right). \tag{2.28}$$

Plugging (2.28) in (2.25) and (2.26), respectively, we get

$$\begin{aligned}
\frac{u^r}{r} = \frac{x_1 u^1 + x_2 u^2}{r^2} &= \frac{x_1^2 + x_2^2}{r^2}\partial_z \Delta^{-1}\left(\frac{\omega_\theta}{r}\right) - 2\frac{x_1^2 + x_2^2}{r^2}\frac{\partial_r}{r}\Delta^{-1}\partial_z \Delta^{-1}\left(\frac{\omega_\theta}{r}\right) \\
&= \partial_z \Delta^{-1}\left(\frac{\omega_\theta}{r}\right) - 2\frac{\partial_r}{r}\Delta^{-1}\partial_z \Delta^{-1}\left(\frac{\omega_\theta}{r}\right).
\end{aligned}$$

Furthermore,

$$\partial_z\left(\frac{u^r}{r}\right) = \partial_z^2 \Delta^{-1}\left(\frac{\omega_\theta}{r}\right) - 2\frac{\partial_r}{r}\Delta^{-1}\partial_z^2 \Delta^{-1}\left(\frac{\omega_\theta}{r}\right).$$

Combining this with Proposition 2.4 ensures us to get the estimate (2.23).

Finally, by using Proposition 2.4, L^p-boundedness of Riesz operator and the Sobolev embedding theorem,

$$\left\|\frac{u^r}{r}\right\|_{L^{\frac{3q}{3-q}}} \leq C\left\|\partial_z \Delta^{-1}\left(\frac{\omega_\theta}{r}\right)\right\|_{L^{\frac{3q}{3-q}}} \leq C\left\|\frac{\omega_\theta}{r}\right\|_{L^q}.$$

This completes the proof. $\square$

Next, we will give a precise expression about $\frac{u^r}{r}$ and $\frac{\omega_\theta}{r}$ by virtue of the algebraic identity (2.22) and the harmonic analysis tools. More precisely:

Proposition 2.6. *Let u be a free divergence axisymmetric vector-field without swirl and $\omega = \mathrm{curl} u$. Then*

$$\frac{u^r}{r} = (c_1 - 4\gamma_1 i)\frac{x_3}{|x|^3} * \frac{\omega_\theta}{r} + 6\gamma_1 i \frac{x_2^2}{r^2}\frac{x_1^2 x_3}{|x|^5} * \frac{\omega_\theta}{r} + 6\gamma_1 i \frac{x_1^2}{r^2}\frac{x_2^2 x_3}{|x|^5} * \frac{\omega_\theta}{r} - 12\gamma_1 i \frac{x_1 x_2}{r^2}\frac{x_1 x_2 x_3}{|x|^5} * \frac{\omega_\theta}{r}, \quad (2.29)$$

where $c_1 = 2\pi^{\frac{3}{2}}\frac{\Gamma(\frac{1}{2})}{\Gamma(1)}$ and $\gamma_1 = i\pi^{\frac{3}{2}}\frac{\Gamma(\frac{1}{2})}{\Gamma(2)}$.

Proof. From the algebraic identities (2.22) and (2.20), we obtain

$$\frac{u^r}{r} = \partial_z \Delta^{-1}\left(\frac{\omega_\theta}{r}\right) - 2\frac{x_2^2}{r^2}\mathcal{R}_{11}\partial_z\Delta^{-1}\left(\frac{\omega_\theta}{r}\right) - 2\frac{x_1^2}{r^2}\mathcal{R}_{22}\partial_z\Delta^{-1}\left(\frac{\omega_\theta}{r}\right) + 4\frac{x_1x_2}{r^2}\mathcal{R}_{12}\partial_z\Delta^{-1}\left(\frac{\omega_\theta}{r}\right). \quad (2.30)$$

On the one hand,

$$\Delta^{-1}\left(\frac{\omega_\theta}{r}\right) = -\mathcal{F}^{-1}\Big(\frac{1}{|\xi|^2}\widehat{\left(\frac{\omega_\theta}{r}\right)}(\xi)\Big) = -c_1\frac{1}{|x|} * \frac{\omega_\theta}{r}.$$

Thus,

$$\partial_z\Delta^{-1}\left(\frac{\omega_\theta}{r}\right) = c_1\frac{x_3}{|x|^3} * \frac{\omega_\theta}{r}.$$

On the other hand,

$$\mathcal{R}_{kj}\partial_z\Delta^{-1}\left(\frac{\omega_\theta}{r}\right) = -i\partial_z\partial_j\mathcal{F}^{-1}\Big(\frac{\xi_k}{|\xi|^4}\widehat{\left(\frac{\omega_\theta}{r}\right)}(\xi)\Big). \quad (2.31)$$

Since ξ_k is a harmonic polynomial of order one, then we can get by using Theorem 5 of Chap-4 in [30] that

$$\mathcal{F}^{-1}\Big(\frac{\xi_k}{|\xi|^4}\Big) = \gamma_1\frac{x_k}{|x|}.$$

Inserting this equality to (2.31), we have

$$\begin{aligned}\mathcal{R}_{kj}\partial_z\Delta^{-1}\left(\frac{\omega_\theta}{r}\right) =& - i\gamma_1\partial_z\partial_j\Big(\frac{x_k}{|x|} * \frac{\omega_\theta}{r}\Big)(x)\\ =& i\gamma_1\delta_{kj}\frac{x_3}{|x|^3} * \frac{\omega_\theta}{r}(x) - 3i\gamma_1\frac{x_kx_jx_3}{|x|^5} * \frac{\omega_\theta}{r}(x).\end{aligned}$$

Plugging these equalities in (2.30) gives the desired result. $\square$

Remark 2.1. Let us point out that the equality (2.29) implies the S-Y estimate of [29]. More precisely, by virtue of (2.29) and the triangle inequality, we can conclude that

$$\begin{aligned}\Big|\frac{u^r}{r}\Big| =& \Big|(c_1 - 4\gamma_1 i)\frac{x_3}{|x|^3} * \frac{\omega_\theta}{r} + 6\gamma_1 i\frac{x_2^2}{r^2}\frac{x_1^2x_3}{|x|^5} * \frac{\omega_\theta}{r} + 6\gamma_1 i\frac{x_1^2}{r^2}\frac{x_2^2x_3}{|x|^5} * \frac{\omega_\theta}{r} - 12\gamma_1 i\frac{x_1x_2}{r^2}\frac{x_1x_2x_3}{|x|^5} * \frac{\omega_\theta}{r}\Big|\\ \le& C\frac{1}{|x|^2} * \Big|\frac{\omega_\theta}{r}\Big|.\end{aligned}$$

Proposition 2.7. *Let u be a smooth axisymmetric vector field with zero divergence and we denote $\omega = \omega_\theta e_\theta$. Then*

$$\Big\|\frac{u^r}{r}\Big\|_{L^\infty} \le C\Big\|\frac{\omega_\theta}{r}\Big\|_{L^2}^{\frac{1}{2}}\Big\|\nabla_h\Big(\frac{\omega_\theta}{r}\Big)\Big\|_{L^2}^{\frac{1}{2}}.$$

Proof. From (2.30), it is clear that

$$\Big\|\frac{u^r}{r}\Big\|_{L^\infty(\mathbb{R}^3)} \leq C\Big\|\partial_z\Delta^{-1}\Big(\frac{\omega_\theta}{r}\Big)\Big\|_{L^\infty(\mathbb{R}^3)} + C\sum_{k,j}^{2}\Big\|\mathcal{R}_{kj}\partial_z\Delta^{-1}\Big(\frac{\omega_\theta}{r}\Big)\Big\|_{L^\infty(\mathbb{R}^3)}. \tag{2.32}$$

For the first term in the right side of (2.32), by using Lemma F.1, we obtain

$$\begin{aligned}\Big\|\partial_z\Delta^{-1}\Big(\frac{\omega_\theta}{r}\Big)\Big\|_{L^\infty(\mathbb{R}^3)} \leq & \Big\|\nabla\partial_z\Delta^{-1}\Big(\frac{\omega_\theta}{r}\Big)\Big\|_{L^2(\mathbb{R}^3)}^{\frac12}\Big\|\nabla_h\nabla\partial_z\Delta^{-1}\Big(\frac{\omega_\theta}{r}\Big)\Big\|_{L^2(\mathbb{R}^3)}^{\frac12}\\ \leq & C\Big\|\frac{\omega_\theta}{r}\Big\|_{L^2(\mathbb{R}^3)}^{\frac12}\Big\|\nabla_h\Big(\frac{\omega_\theta}{r}\Big)\Big\|_{L^2(\mathbb{R}^3)}^{\frac12}.\end{aligned}$$

Using Lemma F.1 again and applying the L^p-boundedness of Riesz operator, the second term can be bounded by

$$\Big\|\nabla\mathcal{R}_{kj}\partial_z\Delta^{-1}\Big(\frac{\omega_\theta}{r}\Big)\Big\|_{L^2(\mathbb{R}^3)}^{\frac12}\Big\|\nabla_h\nabla\mathcal{R}_{kj}\partial_z\Delta^{-1}\Big(\frac{\omega_\theta}{r}\Big)\Big\|_{L^2(\mathbb{R}^3)}^{\frac12} \leq C\Big\|\frac{\omega_\theta}{r}\Big\|_{L^2(\mathbb{R}^3)}^{\frac12}\Big\|\nabla_h\Big(\frac{\omega_\theta}{r}\Big)\Big\|_{L^2(\mathbb{R}^3)}^{\frac12}.$$

This ends the proof. □

3 The priori estimate

In this section, we will give some useful priori estimates.

3.1 Energy estimate and higher-order estimate

We start with L^2 energy estimates and the maximum principle.

Proposition 3.1. *Let (u,ρ) be a solution of (1.2), then*

$$\|u(t)\|_{L^2}^2 + \|\nabla_h u\|_{L^2_tL^2}^2 \leq (\|u_0\|_{L^2} + t\,\|\rho_0\|_{L^2})^2 \tag{3.33}$$

and

$$\|\rho(t)\|_{L^2}^2 + \|\nabla_h\rho\|_{L^2_tL^2}^2 \leq \|\rho_0\|_{L^2}^2\,. \tag{3.34}$$

Besides, for $2\leq p\leq\infty$,

$$\|\rho(t)\|_{L^p} \leq \|\rho_0\|_{L^p}\,. \tag{3.35}$$

Proof. The proof is standard, we also give the proof for reader convenience. We first prove the estimate (3.35). Multiplying the second equation of (1.2) by $|\rho|^{p-2}\rho$ and integrating by parts yields that

$$\frac1p\frac{\mathrm{d}}{\mathrm{d}t}\|\rho(t)\|_{L^p}^p + (p-1)\int|\rho|^{p-2}|\nabla_h\rho|^2\mathrm{d}x = 0.$$

Thus we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\rho(t)\|_{L^p}^p \leq 0,$$

which implies immediately

$$\|\rho(t)\|_{L^p} \leq \|\rho_0\|_{L^p}\,.$$

For $p=\infty$, it is just the maximum principle.

For the first one we take the L^2-inner product of the velocity equation with u. From integration by parts and the fact that u is divergence free, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_{L^2}^2 + \|\nabla_h u(t)\|_{L^2}^2 \le \|u(t)\|_{L^2}\|\rho(t)\|_{L^2}. \tag{3.36}$$

Furthermore, we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_{L^2} \le \|\rho(t)\|_{L^2}.$$

By integration in time, we get that

$$\|u(t)\|_{L^2} \le \|u_0\|_{L^2} + \int_0^t \|\rho(\tau)\|_{L^2}\mathrm{d}\tau \le \|u_0\|_{L^2} + t\|\rho_0\|_{L^2},$$

where we used the fact $\|\rho(t)\|_{L^2} \le \|\rho_0\|_{L^2}$. Plugging this estimate into (3.36) yields

$$\frac{1}{2}\|u(t)\|_{L^2}^2 + \int_0^t \|\nabla_h u(\tau)\|_{L^2}^2\mathrm{d}\tau \le \frac{1}{2}\|u_0\|_{L^2}^2 + \big(\|u_0\|_{L^2} + t\|\rho_0\|_{L^2}\big)\|\rho_0\|_{L^2}t.$$

This implies the first result.

Finally, by the same argument as in proof of (3.33), we obtain the estimate (3.34). □

Subsequently, we will establish the estimate of the quantities $\frac{\omega_\theta}{r}$ and ω which enable us to get the global existence of axisymmetric system (1.2).

Proposition 3.2. *Assume that $u_0 \in H^1$, with $\frac{\omega_0}{r} \in L^2$ and $\rho_0 \in L^2$. Let (u, ρ) be a smooth axisymmetric solution (u, ρ) of (1.2) without swirl, then we have*

$$\left\|\frac{\omega}{r}(t)\right\|_{L^2}^2 + \int_0^t \left\|\nabla_h\left(\frac{\omega}{r}\right)(\tau)\right\|_{L^2}^2\mathrm{d}\tau \le 2\left(\left\|\frac{\omega_0}{r}\right\|_{L^2} + \|\rho_0\|_2\right)^2,$$

and

$$\|u(t)\|_{H^1}^2 + \int_0^t \|\nabla_h u(\tau)\|_{H^1}^2\mathrm{d}\tau \le C_0 e^{C_0 t},$$

where C_0 depends only on the norm of the initial data.

Proof. According to the equation (1.10), it is clear that $\frac{\omega_\theta}{r}$ satisfies the following equation

$$\big(\partial_t + u\cdot\nabla\big)\frac{\omega_\theta}{r} - \big(\Delta_h + \frac{2}{r}\partial_r\big)\frac{\omega_\theta}{r} = -\frac{\partial_r\rho}{r}\cdot \tag{3.37}$$

Now we recall that $(\partial_t + u\cdot\nabla)\rho - \Delta_h\rho = 0$, which can be rewritten as

$$(\partial_t + u\cdot\nabla)\rho - (\Delta_h + \frac{2}{r}\partial_r)\rho = -\frac{2}{r}\partial_r\rho. \tag{3.38}$$

In view of (3.37) and (3.38), we can set $\Gamma := \frac{\omega_\theta}{r} - \frac{1}{2}\rho$ and then Γ solves the equation

$$(\partial_t + u\cdot\nabla)\Gamma - (\Delta_h + \frac{2}{r}\partial_r)\Gamma = 0.$$

Taking the L^2-inner product with Γ and integrating by parts, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Gamma(t)\|_{L^2}^2 + \|\nabla_h\Gamma(t)\|_{L^2}^2 \le 0,$$

where we used the facts that u is divergence free and $-\int \frac{\partial_r \Gamma}{r}\Gamma \mathrm{d}x \geq 0$. By integration in time, we obtain that

$$\|\Gamma(t)\|_{L^2}^2 + \int_0^t \|\nabla_h \Gamma(\tau)\|_{L^2}^2 \mathrm{d}\tau \leq \|\Gamma_0\|_{L^2}^2.$$

This together with the estimate (3.34) yield that

$$\begin{aligned}
&\Big\|\frac{\omega_\theta}{r}(t)\Big\|_{L^2}^2 + \int_0^t \Big\|\nabla_h\Big(\frac{\omega_\theta}{r}\Big)(\tau)\Big\|_{L^2}^2 \mathrm{d}\tau \\
\leq &\big(\|\Gamma(t)\|_{L^2} + \|\rho(t)\|_{L^2}\big)^2 + \big(\|\nabla_h\Gamma(t)\|_{L^2_tL^2} + \|\nabla_h\rho(t)\|_{L^2_tL^2}\big)^2 \\
\leq &2\big(\|\Gamma_0\|_{L^2} + \|\rho_0\|_{L^2}\big)^2.
\end{aligned}$$

This gives the first claimed estimate.

To prove the second estimate. By taking the L^2-inner product of (1.9) with ω_θ we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega_\theta(t)\|_{L^2}^2 + \|\nabla_h\omega_\theta(t)\|_{L^2}^2 + \Big\|\frac{\omega_\theta}{r}(t)\Big\|_{L^2}^2 = \int_{\mathbb{R}^3} \frac{u^r}{r}\omega_\theta\omega_\theta \mathrm{d}x - \int_{\mathbb{R}^3} \partial_r\rho\omega_\theta \mathrm{d}x.$$

Integrating by parts,

$$\begin{aligned}
\int_{\mathbb{R}^3} \partial_r\rho\omega_\theta \mathrm{d}x = 2\pi\int \partial_r\rho\omega_\theta r\mathrm{d}r\mathrm{d}z = 2\pi\int \rho\partial_r\omega_\theta r\mathrm{d}r\mathrm{d}z + 2\pi\int \rho\omega_\theta \mathrm{d}r\mathrm{d}z \\
= \int_{\mathbb{R}^3} \rho\partial_r\omega_\theta \mathrm{d}x + \int_{\mathbb{R}^3} \rho\frac{\omega_\theta}{r}\mathrm{d}x.
\end{aligned}$$

Thus, by the Hölder inequality, we have that

$$\begin{aligned}
\Big|\int_{\mathbb{R}^3} \partial_r\rho\omega_\theta \mathrm{d}x\Big| \leq &\|\rho\|_{L^2}\big(\|\nabla_h\omega_\theta\|_{L^2} + \|\omega_\theta/r\|_{L^2}\big) \\
\leq &2\|\rho_0\|_{L^2}^2 + \frac{1}{4}\big(\|\nabla_h\omega_\theta\|_{L^2}^2 + \|\omega_\theta/r\|_{L^2}^2\big).
\end{aligned}$$

Next, by virtue of the equality (6.6), Proposition 2.5 and the Young inequality, we obtain that

$$\begin{aligned}
\Big|\int_{\mathbb{R}^3} \frac{u^r}{r}\omega_\theta\omega_\theta \mathrm{d}x\Big| \leq &\Big\|\frac{u^r}{r}\Big\|_{L^6}^{\frac{3}{4}} \Big\|\partial_z\Big(\frac{u^r}{r}\Big)\Big\|_{L^2}^{\frac{1}{4}} \|\omega_\theta\|_2^{\frac{1}{2}} \|\nabla_h\omega_\theta\|_2^{\frac{1}{2}} \|\omega_\theta\|_2 \\
\leq &C\Big\|\frac{\omega_\theta}{r}\Big\|_{L^2} \|\omega_\theta\|_2^{\frac{3}{2}} \|\nabla_h\omega_\theta\|_2^{\frac{1}{2}} \\
\leq &C\Big\|\frac{\omega_\theta}{r}\Big\|_{L^2}^{\frac{4}{3}} \|\omega_\theta\|_2^2 + \frac{1}{4}\|\nabla_h\omega_\theta\|_2^2.
\end{aligned}$$

Collecting these estimates with Proposition 3.1 yield

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\omega_\theta(t)\|_{L^2}^2 + \|\nabla_h\omega_\theta\|_{L^2}^2 + \Big\|\frac{\omega_\theta}{r}\Big\|_{L^2}^2 \lesssim \|\rho_0\|_2^2 + \Big\|\frac{\omega_\theta}{r}\Big\|_{L^2}^{\frac{4}{3}} \|\omega_\theta\|_2^2.$$

Therefore we get by the Gronwall inequality that

$$\begin{aligned}
&\|\omega_\theta(t)\|_{L^2}^2 + \int_0^t \Big(\|\nabla_h\omega_\theta(\tau)\|_{L^2}^2 + \Big\|\frac{\omega_\theta}{r}(\tau)\Big\|_{L^2}^2\Big)\mathrm{d}\tau \\
\leq &Ce^{\int_0^t \|\frac{\omega_\theta}{r}(\tau)\|_{L^2}^{\frac{4}{3}}\mathrm{d}\tau}\Big(\Big\|\frac{\omega_\theta}{r}(0)\Big\|_{L^2} + \|\rho_0\|_{L^2} t\Big).
\end{aligned}$$

Since $\|\omega\|_{L^2} = \|\omega_\theta\|_{L^2}$ and $\|\nabla_h\omega\|_{L^2}^2 = \|\nabla_h\omega_\theta\|_{L^2}^2 + \big\|\frac{\omega_\theta}{r}\big\|_{L^2}^2$. So, we finally obtain that

$$\|\omega(t)\|_{L^2}^2 + \int_0^t \|\nabla_h\omega(\tau)\|_{L^2}^2 \mathrm{d}\tau \leq C_0 e^{C_0 t}.$$

This together with the energy estimates yields the second desired estimate. This ends the proof. $\square$

3.2 One derivative estimate on vertical variable

In the absence of dissipation on vertical variable, we need to establish the following estimate on vertical variable in order to compensate this deficiency.

Proposition 3.3. *Assume that $\partial_z\rho_0 \in L^2$ and $\partial_z\omega_0 \in L^2$, then we have*

$$\|\partial_z\rho(t)\|_{L^2}^2 + \int_0^t \|\nabla_h\partial_z\rho(\tau)\|_{L^2}^2 \,\mathrm{d}\tau \le C_1 e^{\exp C_1 t}, \tag{3.39}$$

and

$$\|\partial_z\omega(t)\|_{L^2}^2 + \int_0^t \|\nabla_h\partial_z\omega(\tau)\|_{L^2}^2 \,\mathrm{d}\tau \le C_2 e^{\exp C_2 t}, \tag{3.40}$$

where C_1 and C_2 depend only on the norm of the initial data ρ_0 and ω_0.

Proof. Applying the operator ∂_z to the second equation of (1.2), we obtain

$$(\partial_t + u\cdot\nabla)\partial_z\rho - \Delta_h\partial_z\rho = -\partial_z u^r\partial_r\rho - \partial_z u^z\partial_z\rho. \tag{3.41}$$

Taking the L^2-inner product of the equation (3.41) with $\partial_z\rho$ and integrating by parts, we get

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_z\rho(t)\|_{L^2}^2 + \|\nabla_h\partial_z\rho(t)\|_{L^2}^2 = -\int \partial_z u^r\partial_r\rho\partial_z\rho \mathrm{d}x - \int \partial_z u^z\partial_z\rho\partial_z\rho \mathrm{d}x\\
=& -\int \partial_z u^r\partial_r\rho\partial_z\rho\mathrm{d}x + \int \frac{u^r}{r}\partial_z\rho\partial_z\rho\mathrm{d}x + \int \partial_r u^r\partial_z\rho\partial_z\rho\mathrm{d}x\\
:=& I + II + III,
\end{aligned}$$

where we used the fact $\mathrm{div}u = \partial_r u^r + \frac{u^r}{r} + \partial_z u^z = 0$.

For the first term I, by using (6.7), the Hölder and the Young inequalities, we have

$$\begin{aligned}
I \le& \|\partial_z u^r\|_{L^2}^{\frac12}\|\nabla_h\partial_z u^r\|_{L^2}^{\frac12}\|\partial_r\rho\|_{L^2}^{\frac12}\|\partial_z\partial_r\rho\|_{L^2}^{\frac12}\|\partial_z\rho\|_{L^2}^{\frac12}\|\nabla_h\partial_z\rho\|_{L^2}^{\frac12}\\
\le& 2\|\partial_z u^r\|_{L^2}\|\nabla_h\partial_z u^r\|_{L^2}\|\partial_r\rho\|_{L^2}\|\partial_z\rho\|_{L^2} + \frac14\|\nabla_h\partial_z\rho\|_{L^2}^2\\
\le& \|\nabla_h\partial_z u^r\|_{L^2}^2 + \|\omega\|_{L^2}^2\|\partial_r\rho\|_{L^2}^2\|\partial_z\rho\|_{L^2}^2 + \frac14\|\nabla_h\partial_z\rho\|_{L^2}^2.
\end{aligned}$$

We now turn to bound the term II, by using (6.6), Proposition 2.5 and the Young inequality, we have

$$\begin{aligned}
II \le& \|u^r/r\|_{L^6}^{\frac34}\|\partial_z(u^r/r)\|_{L^2}^{\frac14}\|\partial_z\rho\|_{L^2}^{\frac12}\|\nabla_h\partial_z\rho\|_{L^2}^{\frac12}\|\partial_z\rho\|_{L^2}\\
\le& C\|\omega_\theta/r\|_{L^2}\|\partial_z\rho\|_{L^2}^{\frac32}\|\nabla_h\partial_z\rho\|_{L^2}^{\frac12} \le C\|\omega_\theta/r\|_{L^2}^{\frac43}\|\partial_z\rho\|_{L^2}^2 + \frac14\|\nabla_h\partial_z\rho\|_{L^2}^2.
\end{aligned}$$

Similarly, the term III can be bounded by

$$\begin{aligned}
&\|\partial_r u^r\|_{L^6}^{\frac34}\|\partial_z\partial_r u^r\|_{L^2}^{\frac14}\|\partial_z\rho\|_{L^2}^{\frac32}\|\nabla_h\partial_z\rho\|_{L^2}^{\frac12}\\
\le& C\|\partial_r\nabla u\|_{L^2}\|\partial_z\rho\|_{L^2}^{\frac32}\|\nabla_h\partial_z\rho\|_{L^2}^{\frac12} \le C\|\nabla_h\omega\|_{L^2}^{\frac43}\|\partial_z\rho\|_{L^2}^2 + \frac14\|\nabla_h\partial_z\rho\|_{L^2}^2.
\end{aligned}$$

Combining these estimates, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_z\rho(t)\|_{L^2}^2 + \|\nabla_h\partial_z\rho(t)\|_{L^2}^2\\
\lesssim& \|\nabla_h\partial_z u^r\|_{L^2}^2 + \|\omega\|_{L^2}^2\|\partial_r\rho\|_{L^2}^2\|\partial_z\rho\|_{L^2}^2 + \|\nabla_h\omega\|_{L^2}^{\frac43}\|\partial_z\rho\|_{L^2}^2 + \left\|\frac{\omega_\theta}{r}\right\|_{L^2}^{\frac43}\|\partial_z\rho\|_{L^2}^2.
\end{aligned}$$

Since $\|\nabla_h \nabla u\|_{L^2} \simeq \|\nabla_h \omega\|_{L^2}$. By the Gronwall inequality and Proposition 3.2, we obtain the first desired result (3.39).

Applying ∂_z to the equation (1.8), we get

$$\partial_t \partial_z \omega + u \cdot \nabla \partial_z \omega - \Delta_h \partial_z \omega = -\partial_{zr}^2 \rho e_\theta + \frac{\partial_z u^r}{r} \omega + \frac{u^r}{r} \partial_z \omega - \partial_z u^r \partial_r \omega - \partial_z u^z \partial_z \omega.$$

Taking the L^2-inner product to the above equation with $\partial_z \omega$ and integrating by parts, we obtain

$$\begin{aligned}
&\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\partial_z \omega(t)\|_{L^2}^2 + \|\nabla_h \partial_z \omega(t)\|_{L^2}^2 \\
=&-\int \partial_{zr}^2 \rho e_\theta \partial_z \omega \mathrm{d}x + \int \frac{\partial_z u^r}{r} \omega \partial_z \omega \mathrm{d}x + \int \frac{u^r}{r} \partial_z \omega \partial_z \omega \mathrm{d}x \\
&-\int \partial_z u^r \partial_r \omega \partial_z \omega \mathrm{d}x - \int \partial_z u^z \partial_z \omega \partial_z \omega \mathrm{d}x \\
=&-\int \partial_{zr}^2 \rho e_\theta \partial_z \omega \mathrm{d}x + \int \partial_z \left(\frac{u^r}{r}\right) \omega \partial_z \omega \mathrm{d}x + 2\int \frac{u^r}{r} \partial_z \omega \partial_z \omega \mathrm{d}x \\
&-\int \partial_z u^r \partial_r \omega \partial_z \omega \mathrm{d}x + \int \partial_r u^r \partial_z \omega \partial_z \omega \mathrm{d}x \\
:=&\sum_{i=1}^{5} J_i.
\end{aligned}$$

Here we used the fact $\mathrm{div} u = \partial_r u^r + \frac{u^r}{r} + \partial_z u^z = 0$.

By the Hölder inequality and the Cauchy-Schwarz inequality, we know

$$J_1 \leq \left\|\partial_{zr}^2 \rho\right\|_{L^2} \|\partial_z \omega\|_{L^2} \leq \|\partial_z \omega\|_{L^2}^2 + \left\|\partial_{zr}^2 \rho\right\|_{L^2}^2.$$

By the inequality (6.6) and Proposition 2.5,

$$\begin{aligned}
J_2 \leq& \left\|\partial_z \left(\frac{u^r}{r}\right)\right\|_{L^2} \|\omega\|_{L^6}^{\frac{3}{4}} \|\partial_z \omega\|_{L^2}^{\frac{1}{4}} \|\partial_z \omega\|_{L^2}^{\frac{1}{2}} \|\nabla_h \partial_z \omega\|_{L^2}^{\frac{1}{2}} \\
\leq& C\left\|\partial_z \left(\frac{u^r}{r}\right)\right\|_{L^2} \|\nabla \omega\|_{L^2}^{\frac{3}{4}} \|\partial_z \omega\|_{L^2}^{\frac{3}{4}} \|\nabla_h \partial_z \omega\|_{L^2}^{\frac{1}{2}} \\
\leq& C\left\|\frac{\omega_\theta}{r}\right\|_{L^2} \|\nabla_h \omega\|_{L^2}^{\frac{3}{4}} \|\partial_z \omega\|_{L^2}^{\frac{3}{4}} \|\nabla_h \partial_z \omega\|_{L^2}^{\frac{1}{2}} + C\left\|\frac{\omega_\theta}{r}\right\|_{L^2} \|\partial_z \omega\|_{L^2}^{\frac{3}{2}} \|\nabla_h \partial_z \omega\|_{L^2}^{\frac{1}{2}} \\
\leq& C\left\|\frac{\omega_\theta}{r}\right\|_{L^2}^{\frac{4}{3}} \|\nabla_h \omega\|_{L^2}^2 + C\left\|\frac{\omega_\theta}{r}\right\|_{L^2}^{\frac{4}{3}} \|\partial_z \omega\|_{L^2}^2 + \frac{1}{8} \|\nabla_h \partial_z \omega\|_{L^2}^2.
\end{aligned}$$

For the third term J_3, we get by virtue of the inequality (6.6) and Proposition 2.5 that

$$\begin{aligned}
J_3 \leq& 2\left\|\frac{u^r}{r}\right\|_{L^6}^{\frac{3}{4}} \left\|\partial_z \left(\frac{u^r}{r}\right)\right\|_{L^2}^{\frac{1}{4}} \|\partial_z \omega\|_{L^2}^{\frac{3}{2}} \|\nabla_h \partial_z \omega\|_{L^2} \\
\leq& C\left\|\frac{\omega}{r}\right\|_{L^2}^{\frac{4}{3}} \|\partial_z \omega\|_{L^2}^2 + \frac{1}{8} \|\nabla_h \partial_z \omega\|_{L^2}^2.
\end{aligned}$$

Arguing as for proving J_2, the term J_4 can be bounded as follows.

$$\begin{aligned}
J_4 \leq& \|\partial_r \omega\|_{L^2}^{\frac{1}{2}} \|\partial_z \partial_r \omega\|_{L^2}^{\frac{1}{2}} \|\partial_z u^r\|_{L^2}^{\frac{1}{2}} \|\nabla_h \partial_z u^r\|_{L^2}^{\frac{1}{2}} \|\partial_z \omega\|_{L^2}^{\frac{1}{2}} \|\nabla_h \partial_z \omega\|_{L^2}^{\frac{1}{2}} \\
\leq& \|\omega\|_{L^2}^{\frac{1}{2}} \|\partial_r \omega\|_{L^2}^{\frac{1}{2}} \|\nabla_h \partial_z u^r\|_{L^2}^{\frac{1}{2}} \|\partial_z \omega\|_{L^2}^{\frac{1}{2}} \|\nabla_h \partial_z \omega\|_{L^2}
\end{aligned}$$

$$
\begin{aligned}
&\leq C \|\omega\|_{L^2} \|\partial_r\omega\|_{L^2} \|\nabla_h\partial_z u^r\|_{L^2} \|\partial_z\omega\|_{L^2} + \frac{1}{8} \|\nabla_h\partial_z\omega\|_{L^2}^2 \\
&\leq C \|\omega\|_{L^2}^2 \|\nabla_h\partial_z u^r\|_{L^2}^2 \|\partial_z\omega\|_{L^2}^2 + C \|\partial_r\omega\|_{L^2}^2 + \frac{1}{8} \|\nabla_h\partial_z\omega\|_{L^2}^2 .
\end{aligned}
$$

We turn to bound the term J_5, by Lemma F.2 and the Young inequality, we obtain

$$
\begin{aligned}
J_5 &\leq \left\|\partial_1 u^1\right\|_{L^2}^{\frac{1}{2}} \left\|\partial_z\partial_1 u^1\right\|_{L^2}^{\frac{1}{2}} \|\partial_z\omega\|_{L^2} \|\nabla_h\partial_z\omega\|_{L^2} \\
&\leq \|\omega\|_{L^2} \left\|\partial_z\partial_1 u^1\right\|_{L^2} \|\partial_z\omega\|_{L^2}^2 + \frac{1}{8} \|\nabla_h\partial_z\omega\|_{L^2} .
\end{aligned}
$$

Putting this all together and using the fact that $\|\nabla_h\nabla u\|_{L^2} \simeq \|\nabla_h\omega\|_{L^2}$, we get

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \|\partial_z\omega(t)\|_{L^2}^2 + \frac{3}{4} \|\nabla_h\partial_z\omega(t)\|_{L^2}^2 \\
&\leq \|\partial_z\omega\|_{L^2}^2 + \left\|\partial_{zr}^2\rho\right\|_{L^2}^2 + C\left\|\frac{\omega_\theta}{r}\right\|_{L^2}^{\frac{4}{3}} \|\nabla_h\omega\|_{L^2}^2 + C\left\|\frac{\omega_\theta}{r}\right\|_{L^2}^2 \|\partial_z\omega\|_{L^2}^2 \\
&\quad + C \|\omega\|_{L^2}^2 \|\nabla_h\partial_z u^r\|_{L^2}^2 \|\partial_z\omega\|_{L^2}^2 + C \|\partial_r\omega\|_{L^2}^2 + \|\omega\|_{L^2} \|\nabla_h\nabla u\|_{L^2} \|\partial_z\omega\|_{L^2}^2 \\
&\lesssim \Big(1 + \left\|\frac{\omega_\theta}{r}\right\|_{L^2}^{\frac{4}{3}} + \|\omega\|_{L^2}^2 \|\nabla_h\omega\|_{L^2}^2\Big) \|\partial_z\omega\|_{L^2}^2 + \left\|\partial_{zr}^2\rho\right\|_{L^2}^2 + \left\|\frac{\omega_\theta}{r}\right\|_{L^2}^{\frac{4}{3}} \|\nabla_h\omega\|_{L^2}^2 + \|\partial_r\omega\|_{L^2}^2 .
\end{aligned}
$$

This together with Proposition 3.2 and the Gronwall inequality yields the desired the result. □

3.3 Strong a priori estimate

In the following, our target is to establish the global estimate about Lipschitz norm of the velocity which ensures the global existence of solution.

Let us first give a useful lemma which provides the maximal smooth effect of the velocity in horizontal direction.

Lemma 3.4. *Let $s_1, s_2 \in \mathbb{R}$ and $p \in [2, \infty[$. Assume that (u, ρ) be a smooth solution of the system* (1.2)*, then there holds that*

$$
\|u\|_{L_t^1 B_{p,1}^{s_1+2,s_2}} \lesssim \|u_0\|_{B_{p,1}^{s_1,s_2}} + \|u\|_{L_t^1 B_{p,1}^{s_1,s_2}} + \|u \otimes u\|_{L_t^1 B_{p,1}^{s_1+1,s_2} \cap L_t^1 B_{p,1}^{s_1,s_2+1}} + \|\rho\|_{L_t^1 B_{p,1}^{s_1,s_2}} . \tag{3.42}
$$

Proof. Applying the operator $\Delta_q^h \Delta_k^v$ to (1.2) and using Duhamel formula we get

$$
u_{q,k}(t) = e^{t\Delta_h} u_{q,k}(0) - \int_0^t e^{(t-\tau)\Delta_h} \Delta_q^h \Delta_k^v \mathcal{P}(u \cdot \nabla u)(\tau, x) \mathrm{d}\tau - \int_0^t e^{(t-\tau)\Delta_h} \Delta_q^h \Delta_k^v \mathcal{P}\rho(\tau, x) e_z \mathrm{d}\tau,
$$

where $u_{q,k} = \Delta_q^h \Delta_k^v u$ and $\mathcal{P}$ is the Leray projection on divergence free vector fields.

According to Proposition 2.3, we have the following estimate for $q \geq 0$

$$
\|e^{t\Delta_h} \Delta_q^h \Delta_k^v f\|_{L^p(\mathbb{R}^3)} \leq \Big\| \|e^{t\Delta_h} \Delta_q^h \Delta_k^v f\|_{L^p(\mathbb{R}_h^2)} \Big\|_{L^p(\mathbb{R}_v)} \leq C e^{-ct2^{2q}} \|\Delta_q^h \Delta_k^v f\|_{L^p(\mathbb{R}^3)}.
$$

Therefore, for $q \geq 0$, we have

$$
\|u_{q,k}^r\|_{L_t^1 L^p} \lesssim 2^{-2q} \|u_{q,k}(0)\|_{L^p} + 2^{-2q} \int_0^t \|\Delta_q^h \Delta_k^v (u \cdot \nabla u)(\tau)\|_{L^p} \mathrm{d}\tau + 2^{-2q} \|\rho_{q,k}\|_{L_t^1 L^p}
$$

Multiplying $2^{q(s_1+2)}2^{ks_2}$ and summing over q,k, we obtain

$$
\begin{aligned}
&\sum_{q=0,k=-1}^{+\infty} 2^{q(s_1+2)}2^{ks_2}\|u_{q,k}\|_{L_t^1L^p}\\
\lesssim &\sum_{q=0,k=-1}^{+\infty} 2^{qs_1}2^{ks_2}\|u_{q,k}(0)\|_{L^p}+\int_0^t\sum_{q=0,k=-1}^{+\infty}2^{q(s_1+1)}2^{ks_2}\|\Delta_q^h\Delta_k^v(u\otimes u)(\tau)\|_{L^p}\mathrm{d}\tau\\
&+\int_0^t\sum_{q=0,k=-1}^{+\infty}2^{qs_1}2^{k(s_2+1)}\|\Delta_q^h\Delta_k^v(u\otimes u)(\tau)\|_{L^p}\mathrm{d}\tau+\int_0^t\sum_{q=0,k=-1}^{+\infty}2^{qs_1}2^{ks_2}\|\rho_{q,k}(\tau)\|_{L^p}\mathrm{d}\tau
\end{aligned}
$$

It follows that,

$$
\begin{aligned}
\|u\|_{L_t^1B_{p,1}^{s_1+2,s_2}}\lesssim&\|u\|_{L_t^1B_{p,1}^{s_1,s_2}}+\sum_{q=0,k=-1}^{+\infty}2^{q(s_1+2)}2^{ks_2}\|u_{q,k}\|_{L_t^1L^p}\\
\lesssim&\|u\|_{L_t^1B_{p,1}^{s_1,s_2}}+\|u_0\|_{B_{p,1}^{s_1,s_2}}+\|u\otimes u\|_{L_t^1B_{p,1}^{s_1+1,s_2}}+\|u\otimes u\|_{L_t^1B_{p,1}^{s_1,s_2+1}}+\|\rho\|_{L_t^1B_{p,1}^{s_1,s_2}}.
\end{aligned}
$$

This ends the proof. $\square$

Proposition 3.5. *Let $u_0\in H^1$ be a divergence free axisymmetric without swirl vector field such that $\frac{\omega_0}{r}\in L^2$, $\partial_z\omega_0\in L^2$ and $\rho_0\in H^{0,1}$ an axisymmetric function. Then any smooth solution (u,ρ) of the system* (1.2) *satisfies*

$$\|\nabla u\|_{L_t^1L^\infty}\le C_0e^{\exp C_0t}.$$

Here, the constant C_0 depends on the initial data.

Proof. According to the structure of axisymmetric flows and the incompressible property of velocity, we know that $\mathrm{div}u=\partial_ru^r+\frac{u^r}{r}+\partial_zu^z=0$ and $\omega_\theta=\partial_zu^r-\partial_ru^z$. Therefore

$$
\begin{aligned}
\|\nabla u\|_{L_t^1L^\infty}\le&\|\partial_ru^r\|_{L_t^1L^\infty}+\|\partial_zu^r\|_{L_t^1L^\infty}+\|\partial_ru^z\|_{L_t^1L^\infty}+\|\partial_zu^z\|_{L_t^1L^\infty}\\
\lesssim&\Big\|\frac{u^r}{r}\Big\|_{L_t^1L^\infty}+\|\partial_ru^r\|_{L_t^1L^\infty}+\|\partial_zu^r\|_{L_t^1L^\infty}+\|\partial_ru^z\|_{L_t^1L^\infty}.
\end{aligned}
\tag{3.43}
$$

For the quantity $\|\frac{u^r}{r}\|_{L_t^1L^\infty}$. By virtue of Proposition 2.7 and Proposition 3.2, we get that

$$\Big\|\frac{u^r}{r}\Big\|_{L_t^1L^\infty}\le C\Big\|\frac{\omega_\theta}{r}\Big\|_{L_t^\infty L^2}^{\frac12}\Big\|\nabla_h\Big(\frac{\omega_\theta}{r}\Big)\Big\|_{L_t^1L^2}^{\frac12}\le Ce^{Ct}.$$

Next, we turn to bound the quantity $\|\partial_zu^r\|_{L_t^1L^\infty}$, by using Lemma F.1 and the Bernstein inequality, we have

$$\|\partial_zu^r\|_{L_t^1L^\infty}\le C\int_0^t\|\partial_z\nabla u^r\|_{L^2}^{\frac12}\|\nabla_h\partial_z\nabla u^r\|_{L^2}^{\frac12}\mathrm{d}\tau\le C\|\partial_z\omega\|_{L_t^\infty L^2}^{\frac12}\|\nabla_h\partial_z\omega\|_{L_t^1L^2}^{\frac12}.$$

For the quantity $\|\partial_ru^r\|_{L_t^1L^\infty}$ and $\|\partial_ru^z\|_{L_t^1L^\infty}$, by taking advantage of Lemma 2.1, we know

$$\|\partial_ru^r\|_{L_t^1L^\infty}+\|\partial_ru^z\|_{L_t^1L^\infty}\le C\,\|u\|_{L_t^1B_{2,1}^{2,\frac12}}.$$

Furthermore, by virtue of Lemma 3.4, we get

$$\|u\|_{L_t^1B_{2,1}^{2,\frac12}}\lesssim\|u_0\|_{B_{2,1}^{0,\frac12}}+\|u\|_{L_t^1B_{2,1}^{0,\frac12}}+\|u\otimes u\|_{L_t^1B_{2,1}^{1,\frac12}}+\|u\otimes u\|_{L_t^1B_{2,1}^{0,\frac32}}+\|\rho\|_{L_t^1B_{2,1}^{0,\frac12}}$$

$$\lesssim \|u_0\|_{H^{1,1}} + \|u\|_{L^1_t H^{1,1}} + \|u\otimes u\|_{L^1_t H^{2,1}} + \|u\otimes u\|_{L^1_t H^{\frac54,\frac74}} + \|\rho\|_{L^1_t H^{1,1}}$$
$$\lesssim \|u_0\|_{H^{1,1}} + \|u\|_{L^1_t H^{1,1}} + \|u\|^2_{L^2_t H^{2,1}} + \|u\|^2_{L^2_t H^{\frac54,\frac74}} + \|\rho\|_{L^1_t H^{1,1}}\,.$$

On the other hand, from the definition of space, we have

$$\|u\|_{L^2_t H^{2,1}} \lesssim \|u\|_{L^2_t L^2} + \|\partial_z u\|_{L^2_t L^2} + \left\|\nabla_h^2 u\right\|_{L^2_t L^2} + \left\|\nabla_h^2 \partial_z u\right\|_{L^2_t L^2}$$

and

$$\begin{aligned}\|u\|_{L^2_t H^{\frac54,\frac74}} &\lesssim \|u\|_{L^2_t L^2} + \left\|\Lambda_h^{\frac54} u\right\|_{L^2_t L^2} + \left\|\Lambda_v^{\frac74} u\right\|_{L^2_t L^2} + \left\|\Lambda_h^{\frac54}\Lambda_v^{\frac74} u\right\|_{L^2_t L^2}\\ &\lesssim \|u\|_{L^2_t L^2} + \left\|\nabla_h u\right\|_{L^2_t L^2} + \left\|\nabla_h^2 u\right\|_{L^2_t L^2} + \left\|\partial_z u\right\|_{L^2_t L^2} + \left\|\partial_z^2 u\right\|_{L^2_t L^2} + \left\|\nabla_h\partial_z \omega\right\|_{L^2_t L^2}.\end{aligned}$$

It remains to bound the norm of ρ. By the first estimate of Proposition 3.2 and Proposition 3.3, we have

$$\|\rho\|_{L^1_t H^{1,1}} \lesssim \left\|\rho\right\|_{L^1_t L^2} + \left\|\partial_z\rho\right\|_{L^1_t L^2} + \left\|\nabla_h\rho\right\|_{L^1_t L^2} + \left\|\nabla_h\partial_z\rho\right\|_{L^1_t L^2} \le Ce^{\exp Ct}.$$

Collecting these estimates with Proposition 3.33, Proposition 3.2 and Proposition 3.3 yields that

$$\|\nabla u\|_{L^1_t L^\infty} \le C_0 e^{\exp C_0 t}.$$

This ends the proof. □

4 Proof of Theorem 1.1

Here we use the Friedrichs method (see [17] for more details): For $n\ge 1$, let J_n be the spectral cut-off defined by

$$\widehat{J_n f}(\xi) = 1_{[0,n]}(|\xi|)\widehat{f}(\xi), \quad \xi\in\mathbb{R}^3.$$

We consider the following system in the spaces $L^2_n := \{f\in L^2(\mathbb{R}^3)|\ \mathrm{supp} f\subset B(0,n)\}$:

$$\begin{cases}\partial_t u + \mathcal{P}J_n \mathrm{div}(\mathcal{P}J_n u\otimes \mathcal{P}J_n u) - \Delta_h \mathcal{P}J_n u = \mathcal{P}J_n(\rho e_3),\\ \partial_t\rho + J_n\mathrm{div}(J_n u J_n\rho) - \Delta_h J_n\rho = 0,\\ (\rho,u)|_{t=0} = J_n(\rho_0,u_0).\end{cases} \tag{4.44}$$

The Cauchy-Lipschitz theorem entails that this system exists a unique maximal solution (ρ_n,u_n) in $\mathcal{C}^1([0,T_n^*[;L^2_n)$. On the other hand, we observe that $J_n^2 = J_n, \mathcal{P}^2 = \mathcal{P}$ and $J_n\mathcal{P} = \mathcal{P}J_n$. It follows that $(\rho_n,\mathcal{P}u_n)$ and $(J_n\rho_n, J_n\mathcal{P}u_n)$ are also solutions. The uniqueness gives that $\mathcal{P}u_n = u_n, J_n u_n = u_n$ and $J_n\rho_n = \rho_n$. Therefore

$$\begin{cases}\partial_t u_n + \mathcal{P}J_n\mathrm{div}(u_n\otimes u_n) - \Delta_h u_n = \mathcal{P}J_n(\rho_n e_3),\\ \partial_t\rho_n + J_n\mathrm{div}(u_n\rho_n) - \Delta_h\rho_n = 0,\\ \mathrm{div}u_n = 0,\\ (\rho_n,u_n)|_{t=0} = J_n(\rho_0,u_0).\end{cases} \tag{4.45}$$

As the operators J_n and $\mathcal{P}J_n$ are the orthogonal projectors for the L^2-inner product, the above formal calculations remain unchanged. We will start with the following stability results.

Lemma 4.1. *Let u_0 be a free divergence axisymmetric vector-field without swirl and ρ_0 an axisymmetric scalar function. Then*

(*i*) *for every $n \in \mathbb{N}$, $u_{0,n}$ and $\rho_{0,n}$ are axisymmetric and* $\mathrm{div} u_{0,n} = 0$.

(*ii*) *If $u_0 \in H^1$ is such that $(\text{curl } u_0)/r \in L^2$ and $\rho_0 \in H^{0,1}$. Then there exists a constant C independent of n such that*

$$\|u_{0,n}\|_{H^1} \leq \|u_0\|_{H^1}, \quad \big\|(\text{curl } u_{0,n})/r\big\|_{L^2} \leq C\big\|(\text{curl } u_0)/r\big\|_{L^2},$$
$$\|\rho_{0,n}\|_{L^2} \leq \|\rho_0\|_{L^2}, \quad \|\rho_{0,n}\|_{H^{0,1}} \leq C\|\rho_0\|_{H^{0,1}}.$$

Proof. The proof of $\big\|(\text{curl } u_{0,n})/r\big\|_{L^2} \leq C\big\|(\text{curl } u_0)/r\big\|_{L^2}$ is subtle, one can see [16] for more details. Other estimates can be proved by the standard methods. □

Now, we come back to the proof of the existence parts of Theorem 1.1. From Lemma 4.1, we observe that the initial structure of axisymmetry is preserved for every n and the involved norms are uniformly controlled with respect to this parameter n. This ensures us to construct locally in time a unique solution (u_n, ρ_n) to the approximate system (4.45). On the other hand, we have seen in Proposition 3.5 that the Lipschitz norm of the velocity keeps bounded in finite time. Therefore, this solution is globally defined. By standard compactness arguments and Lions-Aubin Lemma we can show that this family $(u_n, \rho_n)_{n\in\mathbb{N}}$ converges to (u, ρ) which satisfies in turn our initial problem. And the Fatou Lemma ensures $(u, \rho) \in \mathcal{X}$, where

$$\begin{aligned}\mathcal{X} :=& \big(L^\infty_{\text{loc}}(\mathbb{R}_+; H^1) \cap L^2_{\text{loc}}(\mathbb{R}_+; H^{2,1}) \cap L^\infty_{\text{loc}}(\mathbb{R}_+; H^{1,1} \cap H^{0,2}) \cap L^2_{\text{loc}}(\mathbb{R}_+; H^{2,1} \cap H^{1,2})\\ &\cap L^1_{\text{loc}}(\mathbb{R}_+; \text{Lip})\big) \times \big(L^\infty_{\text{loc}}(\mathbb{R}_+; H^{0,1}) \cap L^2_{\text{loc}}(\mathbb{R}_+; H^{1,1})\big).\end{aligned}$$

It remains to prove the time continuity of the solution (u, ρ). We only show that u belongs to $\mathcal{C}(\mathbb{R}_+; H^1)$, the other terms can be treated the same way. First we show the continuity of u in H^1. Indeed, we just need to show that $\omega \in \mathcal{C}(\mathbb{R}_+; L^2)$. Let us recall the vorticity equation

$$\partial_t \omega + u \cdot \nabla \omega - \Delta_h \omega = -\partial_r \rho e_\theta + \frac{u^r}{r}\omega.$$

It is easy to check that the source terms belong to $L^2_{\text{loc}}(\mathbb{R}_+; L^2)$. Using the fact $\nabla u \in L^1_{\text{loc}}(\mathbb{R}_+; \text{Lip})$ and applying Proposition F.4, we get the desired result $\omega \in \mathcal{C}(\mathbb{R}_+; L^2)$.

Next, let us turn to prove the uniqueness. We assume that $(u_i, \rho_i) \in \mathcal{X}, 1 \leq i \leq 2$ be two solutions of the system (1.2) with the same initial data (u_0, ρ_0). Then the difference $(\delta\rho, \delta u, \delta p)$ between two solutions (ρ_1, u_1, p_1) and (ρ_2, u_2, p_2) satisfies

$$\begin{cases}\partial_t \delta u + \mathrm{div}(u_2 \otimes \delta u) - \Delta_h \delta u + \nabla \delta p = -\delta u \cdot \nabla u_1 + \delta \rho e_z,\\ \partial_t \delta\rho + \mathrm{div}(u_2 \delta\rho) - \Delta_h \delta\rho = -\delta u \cdot \nabla \rho_1.\end{cases} \tag{4.46}$$

Taking the L^2-inner product to the first equation of (4.46) with u, we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\delta u(t)\|^2_{L^2} + \|\nabla_h \delta u\|^2_{L^2} =& -\int \delta u \nabla u_1 \delta u \mathrm{d}x + \int \delta\rho e_z \delta u \mathrm{d}x\\ \leq& \|\nabla u_1\|_{L^\infty}\|\delta u\|^2_{L^2} + \|\delta\rho\|_{L^2}\|\delta u\|_{L^2}.\end{aligned} \tag{4.47}$$

On the other hand, by the same computation, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\delta\rho(t)\|^2_{L^2} + \|\nabla_h \delta\rho\|^2_{L^2} = -\int \delta u \nabla \rho_1 \delta\rho \mathrm{d}x = -\int (\delta u)^r \partial_r \rho_1 \delta\rho \mathrm{d}x - \int (\delta u)^z \partial_z \rho_1 \delta\rho \mathrm{d}x.$$

By Lemma F.3 and the Young inequality,

$$\begin{aligned}\int (\delta u)^r \partial_r \rho_1 \delta\rho \mathrm{d}x \leq & \|(\delta u)^r\|_{L^2}^{\frac{1}{2}} \|\nabla_h (\delta u)^r\|_{L^2}^{\frac{1}{2}} \|\partial_r \rho_1\|_{L^2}^{\frac{1}{2}} \|\partial_z \partial_r \rho_1\|_{L^2}^{\frac{1}{2}} \|\delta\rho\|_{L^2}^{\frac{1}{2}} \|\nabla_h \delta\rho\|_{L^2}^{\frac{1}{2}} \\ \leq & C \|\partial_r \rho_1\|_{L^2} \|\partial_z \partial_r \rho_1\|_{L^2} \|\delta u\|_{L^2} \|\delta\rho\|_{L^2} + \frac{1}{2} \|\nabla_h \delta u\|_{L^2} \|\nabla_h \delta\rho\|_{L^2} .\end{aligned}$$

Using Lemma F.3 and $\mathrm{div}\delta u = 0$, we have

$$\begin{aligned}\int (\delta u)^z \partial_z \rho_1 \delta\rho \mathrm{d}x \leq & \|(\delta u)^z\|_{L^2}^{\frac{1}{2}} \|\partial_z (\delta u)^z\|_{L^2}^{\frac{1}{2}} \|\partial_z \rho_1\|_{L^2}^{\frac{1}{2}} \|\nabla_h \partial_z \rho_1\|_{L^2}^{\frac{1}{2}} \|\delta\rho\|_{L^2}^{\frac{1}{2}} \|\nabla_h \delta\rho\|_{L^2}^{\frac{1}{2}} \\ \leq & \|(\delta u)^z\|_{L^2}^{\frac{1}{2}} \|\nabla_h (\delta u)\|_{L^2}^{\frac{1}{2}} \|\partial_z \rho_1\|_{L^2}^{\frac{1}{2}} \|\nabla_h \partial_z \rho_1\|_{L^2}^{\frac{1}{2}} \|\delta\rho\|_{L^2}^{\frac{1}{2}} \|\nabla_h \delta\rho\|_{L^2}^{\frac{1}{2}} \\ \leq & C \|\partial_z \rho_1\|_{L^2} \|\nabla_h \partial_z \rho_1\|_{L^2} \|\delta u\|_{L^2} \|\delta\rho\|_{L^2} + \frac{1}{2} \|\nabla_h \delta u\|_{L^2} \|\nabla_h \delta\rho\|_{L^2} .\end{aligned}$$

The combination of these estimates yield

$$\begin{aligned}& \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\delta\rho(t)\|_{L^2}^2 + \|\nabla_h \delta\rho\|_{L^2}^2 \\ \leq & C(\|\nabla_h \rho_1\|_{L^2} + \|\partial_z \rho_1\|_{L^2}) \|\nabla_h \partial_z \rho_1\|_{L^2} \|\delta u\|_{L^2} \|\delta\rho\|_{L^2} + \|\nabla_h \delta u\|_{L^2} \|\nabla_h \delta\rho\|_{L^2} .\end{aligned}$$

This together with (4.47) yields that

$$\begin{aligned}& \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \Big(\|\delta\rho(t)\|_{L^2}^2 + \|\delta u(t)\|_{L^2}^2 \Big) + \|\nabla_h \delta\rho\|_{L^2}^2 + \|\nabla_h \delta u\|_{L^2}^2 \\ \leq & C\big(\|\nabla_h \rho_1\|_{L^2} + \|\partial_z \rho_1\|_{L^2} \big) \|\nabla_h \partial_z \rho_1\|_{L^2} \|\delta u\|_{L^2} \|\delta\rho\|_{L^2} + \|\nabla_h \delta u\|_{L^2} \|\nabla_h \delta\rho\|_{L^2} \\ & + \|\nabla u_1\|_{L^\infty} \|\delta u\|_{L^2}^2 + \|\delta\rho\|_{L^2} \|\delta u\|_{L^2} .\end{aligned}$$

Consequently,

$$\frac{\mathrm{d}}{\mathrm{d}t} \Big(\|\delta\rho(t)\|_{L^2}^2 + \|\delta u(t)\|_{L^2}^2 \Big) \leq CF(t) \Big(\|\delta\rho(t)\|_{L^2}^2 + \|\delta u(t)\|_{L^2}^2 \Big) ,$$

where

$$F(t) = (\|\nabla_h \rho_1\|_{L^2} + \|\partial_z \rho_1\|_{L^2}) \|\nabla_h \partial_z \rho_1\|_{L^2} + \|\nabla u_1\|_{L^\infty} + 1.$$

By Proposition 3.1 and Proposition 3.3, we know that $F(t)$ is integrable. Therefore, we obtain the uniqueness by using the Gronwall inequality.

5 Proof of Theorem 1.2

In this section, we intend to prove the global existence and the uniqueness of Theorem 1.2 for another class of initial data.

Proposition 5.1. *Assume that $u_0 \in H^1$, with $\frac{\omega_0}{r} \in L^2$ and $\omega_0 \in L^\infty$. Let $\rho_0 \in H^{0,1}$. Then any smooth axisymmetric solution (u, ρ) of (1.2) without swirl satisfies*

$$\|\nabla u(t)\|_{L} \leq Ce^{\exp Ct}\big(\|\omega_0\|_{L^2 \cap L^\infty} + 1\big).$$

Here constant C depends on the initial data.

Proof. Multiplying the vorticity equation (1.9) with $|\omega_\theta|^{p-2}\omega_\theta$ and performing integration in space, we get

$$
\begin{aligned}
&\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int|\omega_\theta|^p\mathrm{d}x+(p-1)\int|\nabla_h\omega_\theta|^2|\omega_\theta|^{p-2}\mathrm{d}x+\int|\omega_\theta|^{p-2}\frac{\omega_\theta^2}{r^2}\mathrm{d}x\\
=&\int\frac{u^r}{r}|\omega_\theta|^p\mathrm{d}x-\int\partial_r\rho|\omega_\theta|^{p-2}\omega_\theta\mathrm{d}x.
\end{aligned}\tag{5.48}
$$

We consider the case $p\geq 4$. For the first term in the last line, we deuce by the Hölder inequality that

$$\int\frac{u^r}{r}|\omega_\theta|^p\mathrm{d}x\leq\Big\|\frac{u^r}{r}\Big\|_{L^\infty}\|\omega_\theta\|_{L^p}^p.$$

Since $\|u\|_{L^{p-2}}\leq\|u\|_{L^2}^{\alpha}\|u\|_{L^p}^{1-\alpha}$ with $\alpha=\frac{4}{(p-2)^2}$ and

$$\int|\omega_\theta|^{p-4}|\nabla_h\omega_\theta|^2\mathrm{d}x=\int|\omega_\theta|^{p-4}|\nabla_h\omega_\theta|^{\frac{2(p-4)}{p-2}}|\nabla_h\omega_\theta|^{\frac{4}{p-2}}\mathrm{d}x\leq\|\nabla_h\omega_\theta\|_{L^2}^{\frac{4}{p-2}}\Big\||\omega_\theta|^{\frac{p-2}{2}}\nabla_h\omega_\theta\Big\|_{L^2}^{\frac{2(p-4)}{p-2}}.$$

By using Lemma F.3 and some based inequalities, the second term can be bounded as follows

$$
\begin{aligned}
&\int\partial_r\rho|\omega_\theta|^{p-2}\omega_\theta\mathrm{d}x\\
\leq&\|\partial_r\rho\|_{L^2}^{\frac12}\|\partial_{rz}^2\rho\|_{L^2}^{\frac12}\Big\|\omega_\theta^{\frac{p-2}{2}}\Big\|_{L^2}^{\frac12}\Big\|\nabla_h\big(\omega_\theta^{\frac{p-2}{2}}\big)\Big\|_{L^2}^{\frac12}\Big\|\omega_\theta^{\frac{p}{2}}\Big\|_{L^2}^{\frac12}\Big\|\nabla_h\big(\omega_\theta^{\frac{p}{2}}\big)\Big\|_{L^2}^{\frac12}\\
\leq&C\sqrt{p}\,\|\partial_r\rho\|_{L^2}^{\frac12}\|\partial_{rz}^2\rho\|_{L^2}^{\frac12}\|\omega_\theta\|_{L^{p-2}}^{\frac{p-2}{4}}\Big\|\omega_\theta^{\frac{p}{2}-2}\nabla_h\omega_\theta\Big\|_{L^2}^{\frac12}\|\omega_\theta\|_{L^p}^{\frac{p}{4}}\Big\|\nabla_h\big(\omega_\theta^{\frac{p}{2}}\big)\Big\|_{L^2}^{\frac12}\\
\leq&Cp^{\frac{1}{p-2}}\|\partial_r\rho\|_{L^2}^{\frac12}\|\partial_{rz}^2\rho\|_{L^2}^{\frac12}\|\omega_\theta\|_{L^2}^{\frac{1}{p-2}}\|\omega_\theta\|_{L^p}^{\frac{p(p-4)}{4(p-2)}}\|\omega_\theta\|_{L^p}^{\frac{p}{4}}\|\nabla_h\omega_\theta\|_{L^2}^{\frac{1}{p-2}}\Big\|\nabla_h\big(\omega_\theta^{\frac{p}{2}}\big)\Big\|_{L^2}^{\frac12+\frac{(p-4)}{2(p-2)}}\\
\leq&Cp^{\frac{2}{p-1}}\|\partial_r\rho\|_{L^2}^{\frac{p-2}{p-1}}\|\partial_{rz}^2\rho\|_{L^2}^{\frac{p-2}{p-1}}\|\nabla_h\omega_\theta\|_{L^2}^{\frac{2}{p-1}}\|\omega_\theta\|_{L^2}^{\frac{2}{p-1}}\|\omega_\theta\|_{L^p}^{p-2-\frac{2}{p-1}}+\frac14\Big\|\nabla_h\big(\omega_\theta^{\frac{p}{2}}\big)\Big\|_{L^2}^2
\end{aligned}
$$

Without loss of generality, we assume that $\|\omega_\theta\|_{L^p}\geq 1$, thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\omega_\theta\|_{L^p}^2\leq C\Big\|\frac{u^r}{r}\Big\|_{L^\infty}\|\omega_\theta\|_{L^p}^2+Cp^{\frac{2}{p-1}}F(t)\leq C\Big\|\frac{u^r}{r}\Big\|_{L^\infty}\|\omega_\theta\|_{L^p}^2+4CF(t),$$

where $F(t):=\|\partial_r\rho\|_{L^2}^{\frac{p-2}{p-1}}\|\partial_{rz}^2\rho\|_{L^2}^{\frac{p-2}{p-1}}\|\nabla_h\omega_\theta\|_{L^2}^{\frac{2}{p-1}}\|\omega_\theta\|_{L^2}^{\frac{2}{p-1}}$. According to Proposition 3.2 and Proposition 3.3, we know that $F(t)$ is integrable. Therefore, by the Gronwall inequality and the relation $\|\omega\|_{L^p}=\|\omega_\theta\|_{L^p}$, we obtain that

$$\|\omega\|_{L^p}^2\leq Ce^{\exp Ct}\Big(\|\omega_0\|_{L^p}^2+\int_0^t F(\tau)\mathrm{d}\tau\Big).$$

This together with the second estimate of Proposition 3.2 yields

$$\|\omega\|_{L^p}^2\leq Ce^{\exp Ct}\quad\text{for}\quad 2\leq p<\infty.$$

Since $\|\nabla u\|_{L^p}\leq C\frac{p^2}{p-1}\|\omega\|_{L^p}$ (see [13]. Chap-3). So, we finally obtain that $\|\nabla u\|_{L}\leq Ce^{\exp Ct}$. This ends the proof. □

Let us first focus on the existence part of Theorem 1.2. Let

$$\mathcal{Y} := \big(L^\infty_{\text{loc}}(\mathbb{R}_+; H^1) \cap L^2_{\text{loc}}(\mathbb{R}_+; H^{1,1}) \cap L^\infty_{\text{loc}}(\mathbb{R}_+; H^{1,1} \cap H^{0,2}) \cap L^2_{\text{loc}}(\mathbb{R}_+; H^{2,1} \cap H^{1,2})$$
$$\cap L^1_{\text{loc}}(\mathbb{R}_+; \mathrm{L})\big) \times \big(L^\infty_{\text{loc}}(\mathbb{R}_+; H^{0,1}) \cap L^2_{\text{loc}}(\mathbb{R}_+; H^{1,1})\big).$$

To prove existence, we smooth out the initial data (u_0, ρ_0) so as to obtain a sequence $(u_{0,n}, \rho_{0,n})_{n\in\mathbb{N}}$ of smooth functions which converges to (u_0, ρ_0). From Lemma 4.1, it is clear that the initial structure of axisymmetry is preserved for every n . By preceding argument, it is easy to check that $(u_n, \rho_n) \in \mathcal{Y}$. Combining this with the equations (1.2), one may conclude that $\partial_t \rho_n \in L^2_{\text{loc}}(\mathbb{R}_+; H^{-1})$ and $\partial_t u_n \in L^2_{\text{loc}}(\mathbb{R}_+; L^2)$. On the other hand, we know that $L^2 \hookrightarrow H^{-1}$ and $H^1 \hookrightarrow L^2$ are locally compact. Therefore, by the classical Aubin-Lions argument and Cantor's diagonal process, we can deduce that, up to extraction, family $(u_n, \rho_n)_{n\in\mathbb{N}}$ has a limit (u, ρ) satisfying the equations (1.2) and that $(u, \rho) \in \mathcal{Y}$. The same arguments as used in Proposition F.4 allows us to show that the time continuity of (u, ρ) in low norms and the weak time continuity. In addition, in a similar way as used in Theorem 5 of [18], we can conclude that $\rho \in \mathcal{C}_b(\mathbb{R}_+; L^2)$.

Now let us turn to the prove the uniqueness. We assume that $(u_i, \rho_i) \in \mathcal{Y}, 1 \leq i \leq 2$ be two solutions of the system (1.2) with the same initial data (u_0, ρ_0). One can write (4.46)

$$\begin{cases} \partial_t \delta u + \operatorname{div}(u_2 \otimes \delta u) + \operatorname{div}(\delta u \otimes u_1) - \Delta_h \delta u + \nabla \delta p = \delta\rho e_z, \\ \partial_t \delta\rho + \operatorname{div}(u_2 \delta\rho) - \Delta_h \delta\rho = -\operatorname{div}(\delta u \rho_1). \end{cases} \tag{5.49}$$

For the sake of convenience, let (α, β, γ) such that $\frac{1}{2} < \alpha < \beta < \gamma \leq 1$. Note that

$$\frac{\|S_q u\|_{L^\infty}}{q} \leq 2^{\frac{3}{q}} \frac{\|S_q u\|_{L^q}}{q} \leq 2^{\frac{3}{2}} \|u\|_{L} .$$

By the same argument as in Proposition F.5 with the vector-field u_2, then there exists T_1 such that

$$\|\delta\rho\|_{H^{\beta-1}} \leq C \int_0^t \|\operatorname{div}(\delta u \rho_1)\|_{H^{\gamma-1}} \, \mathrm{d}\tau \quad \text{for all } t \in [0, T_1].$$

The term on the right side can be bounded as follows. By virtue of the Bony decomposition:

$$\operatorname{div}(\delta u \rho_1) = \operatorname{div}\big(T_{\delta u} \rho_1 + R(\delta u, \rho_1)\big) + \sum_{i=1}^{2} T_{\partial_i \rho_1} \delta u^i, \tag{5.50}$$

where we have used the condition $\operatorname{div}\delta u = 0$.

From standard continuity results for operators T and R (see for example [7]), we have

$$\|T_{\delta u} \rho_1 + R(\delta u, \rho_1)\|_{H^\gamma} \leq C \|\delta u\|_{L^\infty} \|\rho_1\|_{H^\gamma} .$$

As for the last term, since $\gamma - 1 < 0$, we infer that

$$\big\|T_{\partial_i \rho_1} \delta u^i\big\|_{H^{\gamma-1}} \leq C \|\nabla \rho_1\|_{H^{\gamma-1}} \|\delta u\|_{L^\infty} .$$

We eventually get

$$\|\delta\rho\|_{L^\infty_t(H^{\beta-1})} \leq C \|\rho_1\|_{L^2_t(H^\gamma)} \|\delta u\|_{L^2_t L^\infty} . \tag{5.51}$$

Now we turn to bound the term δu. By using Proposition F.5, there exists T_2 such that for all $t \in [0, T_2]$,

$$\|\delta u\|_{L^\infty_t(H^\alpha)} + \|\nabla_h \delta u\|_{L^2_t(H^\alpha)} \leq C\big(\|\delta\rho\|_{L^2_t(H^\beta)} + \|\delta u \cdot \nabla u_1\|_{L^2_t(H^\beta)} \big)$$

for some constant C depending only on α, β and u_2. Using again the Bony decomposition and arguing exactly as for proving (5.51), we get

$$\|\delta u \cdot \nabla u_1\|_{H^{\beta-1}} \leq C \|\delta u\|_{L^\infty} \|u_1\|_{H^\beta}.$$

Therefore, given that $u_1 \in L^\infty_{\rm loc}(\mathbb{R}; H^\beta)$,

$$\|\delta u\|_{L^\infty_t(H^\alpha)} + \|\nabla_h \delta u\|_{L^2_t(H^\alpha)} \leq C(\|\delta\rho\|_{L^2_t(H^{\beta-1})} + \|\delta u\|_{L^2_t(L^\infty)}). \tag{5.52}$$

Next, our task is to show that $\|\delta u\|_{L^2_t L^\infty}$ may be bounded in terms of $\|\delta u\|_{L^\infty_t H^\alpha}$ and of $\|\nabla_h \delta u\|_{L^2_t H^\alpha}$.

According to the assumption $\alpha \in]\frac{1}{2}, 1]$, we have (see the proof in Appendix A)

$$\|\delta u\|_{L^\infty(\mathbb{R}^3)} \leq C \|\delta u\|^{\alpha-\frac{1}{2}}_{H^\alpha(\mathbb{R}^3)} \|\nabla_h \delta u\|^{\frac{3}{2}-\alpha}_{H^\alpha(\mathbb{R}^3)}. \tag{5.53}$$

Combining these estimates, we can deduce that for some constant C depending only on $T = \min\{T_1, T_2\}$ and on the norms of (ρ_1, u_1) and (ρ_2, u_2), we have

$$\|\delta\rho\|_{L^\infty_t H^{\beta-1}} \leq C t^{\frac{\alpha}{2}-\frac{1}{4}} \delta U(t), \quad \delta U(t) \leq C\big(t^{\frac{1}{2}} \|\delta\rho\|_{L^\infty_t H^{\beta-1}} + t^{\frac{\alpha}{2}-\frac{1}{4}} \delta U(t)\big)$$

with

$$\delta U(t) := \|\delta u\|_{L^\infty_t H^\alpha} + \|\nabla_h \delta u\|_{L^2_t H^\alpha}.$$

It follows that $\delta u \equiv 0$ (and thus $\delta\rho \equiv 0$) on a suitably small time interval. Finally, let us notice that our assumptions on the solutions ensure that $\delta\rho \in \mathcal{C}([0, T]; H^{\beta-1})$ and $\delta u \in \mathcal{C}([0, T]; H^\alpha)$. Using a classical connectivity argument, it is now easy to get the uniqueness on the whole interval $[0, \infty[$.

A Appendix

In this section, we first give some useful inequalities which have been used throughout the paper.

Lemma F.1. *There exists a constants C such that*

$$\|u\|_{L^\infty(\mathbb{R}^3)} \leq C \|\nabla u\|^{\frac{1}{2}}_{L^2(\mathbb{R}^3)} \|\nabla_h \nabla u\|^{\frac{1}{2}}_{L^2(\mathbb{R}^3)}. \tag{6.1}$$

Proof. By using the interpolation theorem, we get

$$\|u(x_h, \cdot)\|_{L^\infty(\mathbb{R}_v)} \leq C \|u(x_h, \cdot)\|^{\frac{1}{2}}_{L^6(\mathbb{R}_v)} \|\Lambda_v^{\frac{2}{3}} u(x_h, \cdot)\|^{\frac{1}{2}}_{L^2(\mathbb{R}_v)}.$$

This together with the Minkowski inequality and the embedding theorem gives

$$\begin{aligned}
\|u\|_{L^\infty(\mathbb{R}^3)} \leq & \big\| \|u\|_{L^\infty(\mathbb{R}_v)} \big\|_{L^\infty(\mathbb{R}^2_h)} \\
\leq & C \big\| \|u\|^{\frac{1}{2}}_{L^6(\mathbb{R}_v)} \|\Lambda_v^{\frac{2}{3}} u\|^{\frac{1}{2}}_{L^2(\mathbb{R}_v)} \big\|_{L^\infty(\mathbb{R}^2_h)} \\
\leq & \big\| \|\Lambda_v^{\frac{1}{3}} u\|_{L^\infty(\mathbb{R}^2_h)} \big\|^{\frac{1}{2}}_{L^2(\mathbb{R}_v)} \big\| \|\Lambda_v^{\frac{2}{3}} u\|_{L^\infty(\mathbb{R}^2_h)} \big\|^{\frac{1}{2}}_{L^2(\mathbb{R}_v)}.
\end{aligned} \tag{6.2}$$

On the other hand, using again the interpolation theorem and the embedding theorem, we have

$$
\begin{aligned}
\|\Lambda_v^{\frac{1}{3}}u(\cdot,z)\|_{L^\infty(\mathbb{R}_h^2)} \leq & C\|\Lambda_v^{\frac{1}{3}}u(\cdot,z)\|_{L^6(\mathbb{R}_h^2)}^{\frac{2}{3}}\|\Lambda_h^{\frac{5}{3}}\Lambda_v^{\frac{1}{3}}u(\cdot,z)\|_{L^2(\mathbb{R}_h^2)}^{\frac{1}{3}} \\
\leq & C\|\Lambda_h^{\frac{2}{3}}\Lambda_v^{\frac{1}{3}}u(\cdot,z)\|_{L^2(\mathbb{R}_h^2)}^{\frac{2}{3}}\|\Lambda_h^{\frac{5}{3}}\Lambda_v^{\frac{1}{3}}u(\cdot,z)\|_{L^2(\mathbb{R}_h^2)}^{\frac{1}{3}}
\end{aligned}
\tag{6.3}
$$

and

$$
\begin{aligned}
\|\Lambda_v^{\frac{2}{3}}u(\cdot,z)\|_{L^\infty(\mathbb{R}_h^2)} \leq & C\|\Lambda_v^{\frac{2}{3}}u(\cdot,z)\|_{L^3(\mathbb{R}_h^2)}^{\frac{1}{3}}\|\Lambda_h^{\frac{4}{3}}\Lambda_v^{\frac{2}{3}}u(\cdot,z)\|_{L^2(\mathbb{R}_h^2)}^{\frac{2}{3}} \\
\leq & C\|\Lambda_h^{\frac{1}{3}}\Lambda_v^{\frac{2}{3}}u(\cdot,z)\|_{L^2(\mathbb{R}_h^2)}^{\frac{1}{3}}\|\Lambda_h^{\frac{4}{3}}\Lambda_v^{\frac{2}{3}}u(\cdot,z)\|_{L^2(\mathbb{R}_h^2)}^{\frac{2}{3}}.
\end{aligned}
\tag{6.4}
$$

Inserting (6.3) and (6.4) into (6.2), and using the Hölder inequality, we get

$$
\begin{aligned}
\|u\|_{L^\infty(\mathbb{R}^3)} \leq & \Big\|\|\Lambda_h^{\frac{2}{3}}\Lambda_v^{\frac{1}{3}}u\|_{L^2(\mathbb{R}_h^2)}^{\frac{2}{3}}\|\Lambda_h^{\frac{5}{3}}\Lambda_v^{\frac{1}{3}}u\|_{L^2(\mathbb{R}_h^2)}^{\frac{1}{3}}\Big\|_{L^2(\mathbb{R}_v)}^{\frac{1}{2}}\Big\|\|\Lambda_h^{\frac{1}{3}}\Lambda_v^{\frac{2}{3}}u\|_{L^2(\mathbb{R}_h^2)}^{\frac{1}{3}}\|\Lambda_h^{\frac{4}{3}}\Lambda_v^{\frac{2}{3}}u\|_{L^2(\mathbb{R}_h^2)}^{\frac{2}{3}}\Big\|_{L^2(\mathbb{R}_v)}^{\frac{1}{2}} \\
\leq & \|\Lambda_h^{\frac{2}{3}}\Lambda_v^{\frac{1}{3}}u\|_{L^2(\mathbb{R}^3)}^{\frac{1}{3}}\|\Lambda_h^{\frac{5}{3}}\Lambda_v^{\frac{1}{3}}u\|_{L^2(\mathbb{R}^3)}^{\frac{1}{6}}\|\Lambda_h^{\frac{1}{3}}\Lambda_v^{\frac{2}{3}}u\|_{L^2(\mathbb{R}^3)}^{\frac{1}{6}}\|\Lambda_h^{\frac{4}{3}}\Lambda_v^{\frac{2}{3}}u\|_{L^2(\mathbb{R}^3)}^{\frac{1}{3}} \\
\leq & C\,\|\nabla u\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}}\,\|\nabla_h\nabla u\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}}.
\end{aligned}
$$

This completes the proof. $\square$

Lemma F.2. *Let $q\in]2,\infty[$, there holds that*

$$
\int_{\mathbb{R}^3} fgh\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 \leq C\,\|f\|_{L^{2(q-1)}}^{\frac{q-1}{q}}\,\|\partial_{x_1}f\|_{L^2}^{\frac{1}{q}}\,\|g\|_{L^2}^{\frac{q-2}{q}}\,\|\partial_{x_2}g\|_{L^2}^{\frac{1}{q}}\,\|\partial_{x_3}g\|_{L^2}^{\frac{1}{q}}\,\|h\|_{L^2}\,. \tag{6.5}
$$

In particular, if we take $q=4$ in (6.5)*, we have*

$$
\int_{\mathbb{R}^3} fgh\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 \leq C\,\|f\|_{L^6}^{\frac{3}{4}}\,\|\partial_{x_3}f\|_{L^2}^{\frac{1}{4}}\,\|g\|_{L^2}^{\frac{1}{2}}\,\|\nabla_h g\|_{L^2}^{\frac{1}{2}}\,\|h\|_{L^2}\,. \tag{6.6}
$$

Proof. We only just to show the inequality for functions $f,g,h\in C_0^\infty(\mathbb{R}^3)$ and then pass to the limit by virtue of the density argument.

Using some basic inequalities, we have

$$
\begin{aligned}
& \int_{\mathbb{R}^3} fgh\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 \\
\leq & C\int_{\mathbb{R}^2}\left[\max_{x_1}|f|\left(\int_{\mathbb{R}} g^2\mathrm{d}x_1\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}} h^2\mathrm{d}x_1\right)^{\frac{1}{2}}\right]\mathrm{d}x_2\mathrm{d}x_3 \\
\leq & C\left[\int_{\mathbb{R}^2}\max_{x_1}|f|^q\mathrm{d}x_2\mathrm{d}x_3\right]^{\frac{1}{q}}\left[\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}} g^2\mathrm{d}x_1\right)^{\frac{q}{q-2}}\mathrm{d}x_2\mathrm{d}x_3\right]^{\frac{q-2}{2q}}\left(\int_{\mathbb{R}^3} h^2\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\right)^{\frac{1}{2}} \\
\leq & C\left[\int_{\mathbb{R}^2}\int_{\mathbb{R}}|f|^{q-1}|\partial_{x_1 f}|\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\right]^{\frac{1}{q}}\left[\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}} g^2\mathrm{d}x_1\right)^{\frac{q}{q-2}}\mathrm{d}x_2\mathrm{d}x_3\right]^{\frac{q-2}{2q}}\|h\|_{L^2} \\
\leq & C\,\|f\|_{L^{2(q-1)}}^{\frac{q-1}{q}}\,\|\partial_{x_1}f\|_{L^2}^{\frac{1}{q}}\,\|g\|_{L^2}^{\frac{q-2}{q}}\,\|\partial_{x_2}g\|_{L^2}^{\frac{1}{q}}\,\|\partial_{x_3}g\|_{L^2}^{\frac{1}{q}}\,\|h\|_{L^2}\,.
\end{aligned}
$$

Indeed, by imbedding theorem, Hölder's inequality and Plancherel theorem, we obtain that

$$
\begin{aligned}
\big\| \|g\|_{L^{\frac{2q}{q-2}}_{2,3}(\mathbb{R}^2)} \big\|_{L^2_1(\mathbb{R})} \leq & C \big\| \big\| \|\Lambda_2^{\frac{1}{q}} g\|_{L^2_2(\mathbb{R}^1)} \big\|_{L_3^{\frac{2q}{q-2}}(\mathbb{R}^1)} \big\|_{L^2_1(\mathbb{R})} \leq C \big\| \big\| \|\Lambda_2^{\frac{1}{q}} g\|_{L_3^{\frac{2q}{q-2}}(\mathbb{R}^1)} \big\|_{L^2_2(\mathbb{R}^1)} \big\|_{L^2_1(\mathbb{R})} \\
\leq & C \big\| \big\| \|\Lambda_3^{\frac{1}{q}} \Lambda_2^{\frac{1}{q}} g\|_{L^2_3(\mathbb{R}^1)} \big\|_{L^2_2(\mathbb{R}^1)} \big\|_{L^2_1(\mathbb{R})} = C \big\| \Lambda_3^{\frac{1}{q}} \Lambda_2^{\frac{1}{q}} g \big\|_{L^2} \\
= & C \Big(\int_{\mathbb{R}^3} |\xi_2|^{\frac{2}{q}} |\xi_3|^{\frac{2}{q}} \hat{g}^2(\xi) \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3 \Big)^{\frac{1}{2}} \leq \|\hat{g}\|_{L^2}^{\frac{q-2}{q}} \|\xi_2 \hat{g}\|_{L^2}^{\frac{1}{q}} \|\xi_3 \hat{g}\|_{L^2}^{\frac{1}{q}} \\
\leq & C \|g\|_{L^2}^{\frac{q-2}{q}} \|\partial_{x_2} g\|_{L^2}^{\frac{1}{q}} \|\partial_{x_3} g\|_{L^2}^{\frac{1}{q}} .
\end{aligned}
$$

This completes the proof. □

Lemma F.3. *[14] A constant C exists such that*

$$
\int_{\mathbb{R}^3} fgh \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \leq C \|f\|_{L^2}^{\frac{1}{2}} \|\partial_{x_3} f\|_{L^2}^{\frac{1}{2}} \|g\|_{L^2}^{\frac{1}{2}} \|\nabla_h g\|_{L^2}^{\frac{1}{2}} \|h\|_{L^2}^{\frac{1}{2}} \|\nabla_h h\|_{L^2}^{\frac{1}{2}} . \tag{6.7}
$$

Proof of Inequality (5.53). As for $\alpha \in]\frac{1}{2}, \frac{3}{2}[$, by the interpolation theorem and the embedding theorem, we get that for any $z \in \mathbb{R}$

$$
\begin{aligned}
\|u(\cdot,z)\|_{L^\infty_h(\mathbb{R}^2)} \leq & C \|u(\cdot,z)\|^{\alpha-\frac{1}{2}}_{L_h^{\frac{4}{3-2\alpha}}(\mathbb{R}^2)} \left\| \Lambda_h^{\alpha+\frac{1}{2}} u(\cdot,z) \right\|^{\alpha+\frac{1}{2}}_{L^2_h(\mathbb{R}^2)} \\
\leq & C \left\| \Lambda_h^{\alpha-\frac{1}{2}} u(\cdot,z) \right\|^{\alpha-\frac{1}{2}}_{L^2_h(\mathbb{R}^2)} \left\| \Lambda_h^{\alpha+\frac{1}{2}} u(\cdot,z) \right\|^{\frac{3}{2}-\alpha}_{L^2_h(\mathbb{R}^2)} \\
\leq & C \|u(\cdot,z)\|^{\alpha-\frac{1}{2}}_{H^{\alpha-\frac{1}{2}}(\mathbb{R}^2)} \|\nabla_h u(\cdot,z)\|^{\frac{3}{2}-\alpha}_{H^{\alpha-\frac{1}{2}}(\mathbb{R}^2)} .
\end{aligned} \tag{6.8}
$$

On the other hand, by the trace theorem that $H^\alpha(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}_v; H^{\alpha-\frac{1}{2}}(\mathbb{R}^2_h))$,

$$
\|u\|_{L^\infty_v(H^{\alpha-\frac{1}{2}})} \leq C \|u\|_{(H^\alpha)} \quad \text{and} \quad \|\nabla_h u\|_{L^\infty_v(H^{\alpha-\frac{1}{2}})} \leq C \|\nabla_h u\|_{(H^\alpha)} .
$$

Inserting these inequalities in (6.8), we get the desired result (5.53). □

For the sake of completeness, we give an existence result for the anisotropic equations with a convection term, which is similar to the case of the transport equation in [7, 26].

Proposition F.4. *Let $s \geq -1$, $1 \leq p < \infty$ and $1 \leq r \leq \infty$. Assume that $f_0 \in B^s_{p,r}, g \in L^1([0,T]; B^s_{p,r})$, and that u be a divergence free vector-field satisfying $u \in L^\sigma([0,T]; B^{-m}_{\infty,\infty})$ for some $\sigma > 1$ and $m > 0$, and $\nabla u \in L^1([0,T]; L^\infty)$. Then the following equations*

$$
\begin{cases} \partial_t f + u \cdot \nabla f - \Delta_h f = g, \\ f|_{t=0} = f_0 \end{cases} \tag{6.9}
$$

admits a unique solution f in

- *the space $\mathcal{C}([0,T]; B^s_{p,r})$, if $r < \infty$,*
- *the space $\left(\cap_{s'<s} \mathcal{C}([0,T]; B^{s'}_{p,\infty}) \right) \cap \mathcal{C}_w([0,T]; B^s_{p,\infty})$, if $r = \infty$.*

Moreover, we have

$$\|f(t)\|_{B^s_{p,r}} \le e^{C\int_0^t V(\tau)\mathrm{d}\tau}\Big(\|f_0\|_{B^s_{p,r}} + \int_0^t e^{-C\int_0^\tau V(s)\mathrm{d}s}\|g(\tau)\|_{B^s_{p,r}}\,\mathrm{d}\tau\Big). \tag{6.10}$$

Here $V(t) := \|\nabla u(t)\|_{L^\infty}$.

Proof. Without loss of generality, we assume that u and g are defined on $\mathbb{R}\times\mathbb{R}^n$. We first construct the approximate solutions f_n of (6.9) as follows.

$$\begin{cases}\partial_t f_n + u_n\cdot\nabla f_n - \Delta_h f_n = g_n,\\ u_n := \varphi_n *_t S_n u, \quad g_n := \varphi_n *_t S_n g,\\ f_n|_{t=0} = f_{0,n} := S_n f_0,\end{cases} \tag{6.11}$$

where, φ_n denotes a family of mollifiers with respect to t. Thanks to the properties of mollifier and the operator S_j, it is clear that $f_{0,n}\in B^\infty_{p,r}$ and $u_n, g_n \in \mathcal{C}([0,T];B^\infty_{p,r})$ with $B^\infty_{p,r} := \cap_{s\in\mathbb{R}}B^s_{p,r}$. Moreover, $(f_{0,n})_{n\in\mathbb{N}}$ is bounded in $B^s_{p,r}$, $(g_n)_{n\in\mathbb{N}}$ is bounded in $L^1([0,T];B^s_{p,r})$, $(u_n)_{n\in\mathbb{N}}$ is bounded in $L^\sigma([0,T];B^{-m}_{\infty,\infty})$, and $(\nabla u_n)_{n\in\mathbb{N}}$ is bounded in $L^1([0,T];L^\infty)$.

Applying Δ_k to (6.11), we have

$$\begin{cases}\partial_t\Delta_k f_n + u_n\cdot\nabla\Delta_k f_n - \Delta_h\Delta_k f_n = \Delta_k g_n + R^k_n,\\ \Delta_k f_n|_{t=0} = \Delta_k f_{0,n},\end{cases}$$

where $R^k_n := u_n\cdot\nabla\Delta_k f_n - \Delta_k(u\cdot\nabla f_n)$. Note that

$$-\int \Delta_h(\Delta_k f_n)|\Delta_k f_n|^{p-2}\Delta_k f_n \mathrm{d}x \ge 0,$$

it is easy to conclude that

$$\|\Delta_k f_n(t)\|_{L^p} \le \|\Delta_k f_0\|_{L^p} + \int_0^t \|\Delta_k g_n(\tau)\|_{L^p}\,\mathrm{d}\tau + \int_0^t \left\|R^k_n(\tau)\right\|_{L^p}\mathrm{d}\tau. \tag{6.12}$$

This together with the commutator estimate (see e.g. [26], Chap-2)

$$\left\|R^k_n(t)\right\|_{L^p} \le Cc_k(t)2^{-ks}V_n(t)\,\|f_n(t)\|_{B^s_{p,r}} \quad \text{with} \quad \|c_k(t)\|_{l^r} = 1$$

leads to

$$\|f_n(t)\|_{B^s_{p,r}} \le \|f_0\|_{B^s_{p,r}} + \int_0^t \Big(\|g_n(\tau)\|_{B^s_{p,r}} + CV_n(\tau)\,\|f(\tau)\|_{B^s_{p,r}}\Big)\mathrm{d}\tau,$$

where $V_n(t) := \|\nabla u_n(t)\|_{L^\infty}$. Applying the Gronwall inequality, we obtain

$$\|f_n(t)\|_{B^s_{p,r}} \le e^{C\int_0^t V_n(\tau)\mathrm{d}\tau}\Big(\|f_0\|_{B^s_{p,r}} + \int_0^t e^{-C\int_0^\tau V_n(s)\mathrm{d}s}\|g_n(\tau)\|_{B^s_{p,r}}\,\mathrm{d}\tau\Big).$$

In the following, we shall show that, up to an subsequence, the sequence $(f_n)_{n\in\mathbb{N}}$ converges in $\mathcal{D}'(\mathbb{R}_+\times\mathbb{R}^n)$ to a solution f of (6.9) which has the desired regularity properties. First, one may write

$$\partial_t f_n - g_n = -u_n\cdot\nabla f_n + \Delta_h f_n.$$

Hence $u_n \in L^\sigma([0,T];B^{-m}_{\infty,\infty})$ enables us to conclude that $\partial_t f_n - g_n$ is bounded in $L^\sigma([0,T];B^{-M}_{p,\infty})$ for some sufficiently large $M>0$. For the sake of convenience, let

$$\bar f_n(t) := f_n(t) - \int_0^t g_n(\tau)\mathrm{d}\tau.$$

Thanks to the imbedding theorem, one may deduce that $(\bar f_n)_n$ belongs to $\mathcal{C}^\beta([0,T];B^{-M}_{p,\infty})$ with $\beta>0$ and hence uniformly equicontinuous with value in $B^{-M}_{p,\infty}$. Now, let $(\chi_l)_{l\in\mathbb{N}}$ be a sequence of $C_0^\infty(\mathbb{R}^n)$ cut-off functions supported in the ball $B(0,l+1)$ of $\mathbb{R}^n$ and equal to 1 in a neighborhood of $B(0,l)$. On the other hand, by Theorem 2.94 of [7], we know that the map $u\mapsto \chi_l u$ is compact from $B^s_{p,r}$ to $B^{-M}_{p,\infty}$. By using Ascoli's theorem and the Cantor diagonal process, there exists a subsequence which we still denote by $(\bar f_n)_{n\in\mathbb{N}}$ such that, for all $l\in\mathbb{N}$,

$$\chi_l \bar f_n \to_{n\to\infty} \chi_l \bar f \quad \text{in} \quad \mathcal{C}([0,T];B^{-M}_{p,\infty}).$$

It follows that the sequence $(\bar f_n)_{n\in\mathbb{N}}$ converges to some distribution $\bar f$ in $\mathcal{D}'(\mathbb{R}_+\times\mathbb{R}^n)$.

The only problem is to pass to the limit in $\mathcal{D}'(\mathbb{R}_+\times\mathbb{R}^n)$ for the convection term. Let $\psi\in C_0^\infty(\mathbb{R}_+\times\mathbb{R}^n)$ and $l\in\mathbb{N}$ be such that $\mathrm{supp}\psi\subset[0,T]\times B(0,l)$, we have the decomposition

$$\psi u_n\cdot\nabla f_n - \psi u\cdot\nabla f = \psi u_n\cdot\big(\nabla(\chi_l f_n-\chi_l f)\big) - \psi\chi_l(u_n-u)\cdot\nabla f \tag{6.13}$$

Coming back to the uniform estimates of $f_n\in L^\infty([0,T];B^s_{p,r})$, the Fatou properties of Besov space ensures $\bar f$ belong to $L^\infty([0,T];B^s_{p,r})$. By preceding argument, we find that $\chi_l\bar f_n$ tends to $\chi_l\bar f$ in $\mathcal{C}([0,T];B^{s-\varepsilon}_{p,\infty})$ for all $\varepsilon>0$ and $l\in\mathbb{N}$. Therefore, both two terms in the right of (6.13) tend to zero in $L^\infty([0,T];B^{s-1-\varepsilon}_{p,\infty})$. On the other hand, the sequences $(f_{0,n})_{n\in\mathbb{N}},(g_n)_{n\in\mathbb{N}}$ and $(u_n)_{n\in\mathbb{N}}$ converges to f_0,g and u, respectively. So, we finally conclude that $f:=\bar f+\int_0^t g(\tau)\mathrm{d}\tau$ is a solution of (6.9).

It remains to prove that $f\in\mathcal{C}([0,T];B^s_{p,r})$, when $r<\infty$. Making use of uniform estimates of $\bar f_n$, one can deduce that $\partial_t f$ belongs to $L^1([0,T];B^{-M}_{p,\infty})$. Obviously, for fixed k, $\partial_t\Delta_k f$ belongs to $L^1([0,T];L^p)$ so that each $\Delta_k f$ is continuous in time with value in L^p. This implies $S_k f\in\mathcal{C}([0,T];B^s_{p,r})$ for all $k\in\mathbb{N}$. Since

$$\Delta_{k'}(f-S_k f) = \sum_{|k''-k'|\le 1,k''\ge k}\Delta_{k'}(\Delta_{k''}f),$$

then we have

$$\|f-S_k f\|_{B^s_{p,r}} \le C\Big(\sum_{k'\ge k-1} 2^{k'sr}\,\|\Delta_{k'}f\|_{L^p}\Big)^{\frac1r}.$$

By the same argument as in proof of (6.12), one may conclude that

$$\|\Delta_k f(t)\|_{L^p} \le \|\Delta_k f_0\|_{L^p} + \int_0^t \|\Delta_k g(\tau)\|_{L^p}\,\mathrm{d}\tau + C\int_0^t c_k(t)2^{-ks}V(t)\,\|f(\tau)\|_{B^s_{p,r}}\,\mathrm{d}\tau. \tag{6.14}$$

It follows that

$$\begin{aligned}\|f-S_k f\|_{L^\infty_T(B^s_{p,r})} \le & C\Big(\sum_{k'\ge k-1}\big(2^{k's}\,\|\Delta_{k'}f_0\|_{L^p}\big)^r\Big)^{\frac1r}\\ &+C\int_0^T\Big(\sum_{k'\ge k-1}\big(2^{k's}\,\|\Delta_{k'}g(\tau)\|_{L^p}\big)^r\Big)^{\frac1r}\mathrm{d}\tau\end{aligned}$$

$$+ C \|f\|_{L^\infty_T(B^s_{p,r})} \int_0^T \Big(\sum_{k' \geq k-1} c^r_{k'}(\tau) \Big)^{\frac{1}{r}} V(\tau) \mathrm{d}\tau$$

The fact $f_0 \in B^s_{p,r}$ ensures that the first term tends to zero as k goes to infinity. Since $g, V \in L^1_T(B^s_{p,r})$, one conclude that the terms in the integrals also tends to zero for almost every t. This together with the Lebesgue dominated convergence theorem entails $\|f - S_k f\|_{L^\infty_T(B^s_{p,r})}$ tends to zero as k goes to infinity. Thus, we can conclude that f belongs to $\mathcal{C}([0,T]; B^s_{p,r})$.

For the case $r = \infty$, by using the interpolation theorem, we deduce that for any $t_0 \in [0,T]$ and $s' \in]-M, s[$, there exists a constant $\theta \in]0,1[$ depending on s' such that

$$\begin{aligned} \|u(t) - u(t_0)\|_{B^{s'}_{p,\infty}} \leq & \|u(t) - u(t_0)\|^\theta_{B^{-M}_{p,\infty}} \|u(t) - u(t_0)\|^{1-\theta}_{B^s_{p,\infty}} \\ \leq & 2 \|u(t) - u(t_0)\|^\theta_{B^{-M}_{p,\infty}} \|u\|^{1-\theta}_{L^\infty_T(B^s_{p,\infty})}. \end{aligned}$$

This together with the fact $f \in \mathcal{C}([0,T]; B^{-M}_{p,\infty})$ yields $f \in \mathcal{C}([0,T]; B^{s'}_{p,\infty})$ for all $s' < s$. Now, we only need to prove that $f \in \mathcal{C}_w([0,T]; B^s_{p,\infty})$. Indeed, for fixed $\phi \in \mathcal{S}(\mathbb{R}^n)$, the low-high decomposition technique leads to

$$\langle f(t), \phi \rangle = \langle S_k f(t), \phi \rangle + \langle (\mathrm{Id} - S_k) f(t), \phi \rangle = \langle S_k f(t), \phi \rangle + \langle f(t), (\mathrm{Id} - S_k)\phi \rangle.$$

Combining this with $f \in \mathcal{C}([0,T]; B^{s'}_{p,\infty})$ gives that the function $t \mapsto \langle S_k f(t), \phi \rangle$ is continuous. As for the second term, we have

$$|\langle f(t), (\mathrm{Id} - S_k)\phi \rangle| \leq \|f\|_{B^s_{p,\infty}} \|\phi - S_k \phi\|_{B^{-s}_{p',1}}.$$

It follows that $\langle f(t), (\mathrm{Id} - S_j)\phi \rangle$ tends to zero uniformly on $[0,T]$ as k goes to infinity. This means that $f(t) \in \mathcal{C}_w([0,T]; B^s_{p,\infty})$.

Now, we focus on the proof of the uniqueness. Let f_1 and f_2 solve (6.9) with the same initial datum. If we define $\delta f = f_1 - f_2$, then δf solves

$$\partial_t \delta f + u \cdot \nabla \delta f - \Delta_h \delta f = 0.$$

This together with the estimate (6.10) ensures the uniqueness of solution of (6.9). □

The last part of the appendix is devoted to the proof of losing a priori estimate for (1.2) with $\nabla u \in L^1([0,T]; LL)$, where the LogLip space LL is the set of those functions f which belong to $\mathcal{S}'$ and satisfy

$$\|f\|_{LL} := \sup_{2 \leq q < \infty} \frac{\|\nabla S_q f\|_{L^\infty}}{q+1} < \infty. \tag{6.15}$$

This estimate is the cornerstone to the proof of uniqueness in Theorem 1.2. In the sprite of [7, 17], we prove linear losing a priori estimates for the general anisotropic system with convection. More precisely, we have:

Proposition F.5. *Let $s_1 \in [-\frac{1}{2}, 1[$ and assume that $s \in]s_1, 1[$. Let v satisfies the following system*

$$\begin{cases} \partial_t v + u \cdot \nabla v - \Delta_h v + \nabla p = f + g e_3, \\ \mathrm{div} v = \mathrm{div} u = 0 \end{cases} \tag{6.16}$$

with initial data $v_0 \in H^s$ and source terms $f \in L^1([0,T]; H^s)$, $g \in L^2([0,T]; H^{s-1})$. Assume in addition that, for some $h(t) \in L^1[0,T]$ satisfying

$$\|u\|_{LL} \leq h(t). \tag{6.17}$$

Then there exists a constant C such that for any $\lambda > C, T > 0$ and

$$s_t := s - \lambda \int_0^t h(\tau)\mathrm{d}\tau,$$

the following estimate holds

$$\|v(t)\|_{H^{s_t}} + \|\nabla_h v\|_{L^2_t H^{s_t}} \leq C(1+\sqrt{t})\exp\Big(\frac{C}{\lambda}\int_0^t h(\tau)\mathrm{d}\tau\Big)\big(\|\rho_0\|_{H^s} + \|f\|_{L^1_t H^s} + \|g\|_{L^2_t H^{s-1}}\big).$$

Proof. Applying the operator Δ_q to the system (6.16), we find that for all $q \geq -1$, the f_q solves the following equations

$$\partial_t v_q + S_{q-1}u \cdot v_q - \Delta_h v_q + \nabla p_q = f_q + g_q e_3 + F_q(u,v)$$

with $F_q(u,v) = S_{q-1}u \cdot \nabla v_q - \Delta_q(u \cdot \nabla v)$.

Taking the L^2-inner product to the above equation with v_q and using $\mathrm{div} u = 0$, we see that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v_q\|_{L^2}^2 + \|\nabla_h v_q\|_{L^2}^2 = \int f_q v_q \mathrm{d}x + \int g_q v_q^3 \mathrm{d}x + \int F_q(u,v) v_q \mathrm{d}x. \tag{6.18}$$

Assume that $q \geq 0$. Applying the Bernstein and the Young inequalities, we can deduce that

$$\begin{aligned}\int g_q v_q^3 \mathrm{d}x \leq & C2^{-q}\|g_q\|_{L^2}\left\|\nabla v_q^3\right\|_{L^2} \leq \frac{1}{4}\left\|\nabla v_q^3\right\|_{L^2}^2 + C2^{-2q}\|g_q\|_{L^2}^2 \\ \leq & \frac{1}{4}\left\|\nabla_h v_q^3\right\|_{L^2}^2 + \frac{1}{4}\left\|\partial_3 v_q^3\right\|_{L^2}^2 + C2^{-2q}\|g_q\|_{L^2}^2 \\ \leq & \frac{1}{2}\|\nabla_h v_q\|_{L^2}^2 + C2^{-2q}\|g_q\|_{L^2}^2,\end{aligned}$$

in the last line we have used the fact $\mathrm{div} v = 0$.

Integrating the both sides of (6.18) with respect to t, we get for all $q \geq 0$,

$$\|v_q\|_{L^\infty_t L^2}^2 + \|\nabla_h v_q\|_{L^2_t L^2}^2 \leq \|v_q(0)\|_{L^2}^2 + 2\|f_q\|_{L^1_t L^2}^2 + C2^{-2q}\|g_q\|_{L^2_t L^2}^2 + 2\|F_q(u,v)\|_{L^1_t L^2}^2.$$

For $q = -1$, we merely have

$$\|v_{-1}(t)\|_{L^2} \leq \|v_{-1}(0)\|_{L^2} + \int_0^t \big(\|f_{-1}(\tau)\|_{L^2} + \|g_{-1}(\tau)\|_{L^2} + \|F_{-1}(u,v)(\tau)\|_{L^2}\big)\mathrm{d}\tau.$$

On the other hand, by the Bernstein inequality, we know that

$$\|\nabla_h v_{-1}\|_{L^2_t L^2} \leq C t^{\frac{1}{2}} \|v_{-1}\|_{L^\infty_t L^2}.$$

Therefore, for all $q \geq -1$, we have

$$\begin{aligned}&\|v_q\|_{L^\infty_t L^2} + \|\nabla_h v_q\|_{L^2_t L^2} \\ \leq & 2(1+\sqrt{t})\Big(\|v_q(0)\|_{L^2} + \|f_q\|_{L^1_t L^2} + C2^{-q}\|g_q\|_{L^2_t L^2} + \|F_q(u,v)\|_{L^1_t L^2}\Big).\end{aligned} \tag{6.19}$$

From a standard commutator estimate (see e.g. [7], Chap. 2), we know that for all $\varepsilon \in]0, \frac{s+1}{2}[$, $q \geq -1$ and $t \in [0,T]$

$$2^{q(s-\varepsilon)}\|F_q(u,v)(t)\|_{L^2} \leq C c_q (2+q) h(t) \|v(t)\|_{H^{s-\varepsilon}} \quad \text{with } c_q \in l^2 \tag{6.20}$$

for some constant C depending only on s.

Set $s_t := s - \lambda \int_0^t h(\tau)\mathrm{d}\tau$ for $t \in [0,T]$. Collecting (6.19) and (6.20) yields that

$$
\begin{aligned}
2^{(2+q)s_t} \|v_q\|_{L^2} \le 2(1+\sqrt{t})\Big(& 2^{(2+q)s} \|v_q(0)\|_{L^2} 2^{-\eta(2+q)\int_0^t h(\tau)\mathrm{d}\tau} \\
& + \int_0^t 2^{(2+q)s_\tau} \|f_q(\tau)\|_{L^2} 2^{-\eta(2+q)\int_\tau^t h(\tau')\mathrm{d}\tau'} \mathrm{d}\tau \\
& + \Big(\int_0^t 2^{2(2+q)s_\tau} 2^{-2q} \|g_q(\tau)\|_{L^2}^2 2^{-2\lambda(2+q)\int_\tau^t h(\tau')\mathrm{d}\tau'} \mathrm{d}\tau \Big)^{\frac{1}{2}} \\
& + Cc_q(2+q) \int_0^t h(\tau) 2^{-\lambda(2+q)\int_\tau^t h(\tau')\mathrm{d}\tau'} \|f(\tau)\|_{H^{s_\tau}} \mathrm{d}\tau \Big).
\end{aligned} \tag{6.21}
$$

For the last term of (6.21), we observe that

$$
\begin{aligned}
& Cc_q(2+q) \int_0^t h(\tau) 2^{-\lambda(2+q)\int_\tau^t h(\tau')\mathrm{d}\tau'} \|f(\tau)\|_{H^{s_\tau}} \mathrm{d}\tau \\
\le & Cc_q \frac{1}{\lambda \log 2} \int_0^t \mathrm{d} 2^{-\lambda(2+q)\int_\tau^t h(\tau')\mathrm{d}\tau'} \sup_{\tau\in[0,t]} \|f(\tau)\|_{H^{s_\tau}} \\
= & c_q \frac{C}{\lambda \log 2} \big(1 - 2^{-\lambda(2+q)\int_0^t h(\tau)\mathrm{d}\tau}\big) \sup_{\tau\in[0,t]} \|f(\tau)\|_{H^{s_\tau}} .
\end{aligned}
$$

Thus, multiplying 2^{qs_τ} and taking the l^2-norm of both sides of (6.21) over $q \ge -1$, we get

$$
\begin{aligned}
\sup_{\tau\in[0,t]} \|f(\tau)\|_{H^{s_\tau}} \le 2(1+\sqrt{t})\Big(& \|f_0\|_{H^s} + \|f(\tau)\|_{L_t^1 H^{s_\tau}} + \|g(\tau)\|_{L_t^2 H^{s_\tau - 1}} \\
& + \frac{C}{\lambda \log 2} \sup_{\tau\in[0,t]} \|f(\tau)\|_{H^{s_\tau}} \Big).
\end{aligned}
$$

Choosing λ_0 such that $\frac{2C(1+\sqrt{t})}{\lambda_0 \log 2} = \frac{1}{2}$, we get by the Gronwall inequality that for any $\lambda > \lambda_0$,

$$
\sup_{t\in[0,t]} \|f(t)\|_{H^{s_\tau}} \le 2(1+\sqrt{t}) e^{\frac{C}{\lambda}\int_0^t h(\tau)\mathrm{d}\tau} \Big(\|f_0\|_{H^s} + \|f(\tau)\|_{L_t^1 H^{s_\tau}} + \|g(\tau)\|_{L_t^2 H^{s_\tau - 1}} \Big).
$$

This implies the desired result. $\square$

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